# Economic Dynamics and Uncertainty
# Final Exam – Fall 2020

**1. Dynamic Programming** (12 points, each question 3 points)
We have discussed two different numeric methods to solve (infinite horizon) dynamic programming problems. The two methods differed in using discrete versus continuous function approximation.

i) Why do we have to use approximation methods when solving dynamic programming problems numerically?

> **Solution:** We need to numerically store a value function whose shape is generally unkown. We cannot simply store the values of the function at a continuum of points because it would require and infinite amount of memory. Therefore, we need to find methods that store the value function using a reasonable amount of memory, which generally requires approximation.

ii) Explain the general difference between the two solution approaches.

> **Solution:** The general difference
>
> - Discrete function approximation merely stores the values of the value function at a discrete set of gridpoints. When the algorithm requires evaluation at other points it merely looks up the closest stored value (or interpolates between two stored values).
> - Continuous function approximation uses a function basis. It approximates the value function by a set of basis function and stores the basis coefficients (instead of particular values of the function). When the algorithm requires evaluation it can evaluate the value function at any point.

iii) State and explain all the steps in the solution algorithm where the difference matters (and *only* those steps where it matters!).

> **Solution:** Where does the difference in the value function approximation come into play?
>
> 1. When we first set up the state space:

- If we use a discrete approximation we merely have to create a grid on the state space at which we want to store the values (nodes).
- If we use continuous function approximation we have to define the basis we want to use (type and order) and pick a corresponding grid of interpolation points (nodes).

2. When we evaluate the right hand side of the Bellman equation:
   - If we use a discrete approximation we have to find the nearest neighbor of our next period state value for which we have stored the value function. Then we use that value (or an interpolation of two neighbors).
   - If we use continuous function approximation we simply evaluate the approximate value function at the next period state given its representation in terms of the basis functions.
3. Once we have looped over all the nodes:
   - If we use a discrete approximation we already have the new values of the value function stored at the nodes and we can merely proceed to the next iteration.
   - If we use continuous function approximation we have to estimate new basis coefficients before we can proceed to the next iteration.

The previous three are the important once. If answers were incomplete, students can make up by explaining also the following ones:

- When we plot the value function and control rule it makes a slight difference:
  - If we use a discrete approximation we have a fix set of points that the computer program (e.g. Matlab) will interpolate for us.
  - If we use continuous function approximation we can evaluate our approximation at an arbitrarily dense grid and let the computer program do less interpolation.
- When we simulate our paths:
  Unless we stored the optimal controls, we have to evaluate once again the r.h.s. Bellman equation where the two methods differ as described above.

iv) Assume you are using continuous function approximation using Chebychev polynomials.
a) How will you pick your interpolation nodes? Are they equidistant?
b) Explain the trade-off between using a low versus high order of the basis. Explain what might help you in your judgment whether the approximation is sufficiently good.

**Solution:** a) You should pick the Chebychev nodes, which are the zeros of the next higher order Chebychev polynomial (first part of the answer the most important one). No, they are not equidistant, they are clustered closer towards the boundary of the interval (second part of the sentence not required).
b) The trade-off is between quality of the approximation and computation time (or stability, both answers fine). For the remaining part I expect two reasonable throughts. The preferred ones are:

- Check whether increasing the number of basis functions still changes the results (value function, control rule, time paths).
- Check whether reducing the tolerance in the break criterion still changes the result.

Other accepted arguments include:

- A sign of a bad approximation using Chebychevs can be that the value function or the control rule are wiggly.
- Evaluating the r.h.s. Bellman equation on a grid substantially finer than the nodes and checking the deviation from and evaluation of the approximated value function on that finer grid.

2. **Asset Pricing** (12 points, each question 3 points)

i) Critically assess the following statement: "The Sharpe ratio can be interpreted as the premium above the market price of non-systemic risk." Please explain your answer carefully.

**Solution:** The statement is wrong (confused). The Sharpe ratio of an asset (or portfolio) is its excess return (or risk premium) per unit of risk ($\frac{\mathbb{E}\,\tilde{R}_i - R_f}{\sigma_{R_i}}$). It can be interpreted as the market price of systematic risk. Thus, it is not the excess premium or premium *above* the market price of risk.

ii) Discuss the sign of the Sharpe ratio.

**Solution:** An individual assets Sharpe ratio is positive if the asset adds to market risk, but it can also be negative if the asset's return is negatively correlated to that of the market portfolio.

Notes: The Sharpe ratio of the market portfolio should not be negative. If the Sharpe ratio of an individual stock or some portfolio is negative, it is probably inefficient to hold these stocks outside of the market portfolio. As part of the market portfolio they would reduce the risk but by themselves they are risky.

iii) Assume an asset's volatility is $\sigma_{R_i}$. In the CAPM model, would you expect the asset's risk premium (excess return) to be proportional to $\sigma_{R_i}$? If so, what can you say about the proportionality factor? If not, why not?

**Solution:** Yes and no will both be considered correct if they come with the right explanation. Background (and the students have it on their slides...), in the CAPM we can show that

$$\bar{R}_i - R_f = \rho_{i,M}\ \sigma_{R,i}\ \frac{\bar{R}_M - R_f}{\sigma_M} = \rho_{i,M}\ \sigma_{R,i}\ S_e$$

where $\rho_{i,M}$ is the correlation between asset $i$'s return and that of the market portfolio's return. Thus I would take these answers:

- Yes, the proportionality factors are the market portfolio's Sharpe ratio and the correlation between the correlation coefficient between asset and market return.
- No. Only the part of the asset's risk that is not diversified away when the asset is held as part of the market portfolio is priced. We can split the asset's riskiness into a part proportional to market risk (systematic risk) and a part that is orthogonal to the market risk (diversifiable risk). The excess return is proportional only to the systematic part of the risk.

iv) Discuss the relation betweeen an asset's Sharpe ratio, the volatility of the stochastic discount factor, and the risk-free return factor.

**Solution:** An asset's or portfolio's (absolute) Sharpe ratio is bounded by the product of the stochastic discount factor's volatility and the risk-free rate's return factor. This relation is called the Hansen-Jagannathan bound. It is generally not satisfied for consumption-based asset pricing (based on the standard economic model).
Note: Formula (not expected).

$$\left|\frac{\mathbb{E}\,\tilde{R}_i - R_f}{\sigma_{R_i}}\right| \leq \sigma_M R_f. \tag{1}$$

**3. Consumption, investment, and Epstein-Zin-Weil preferences**
(8 points, each question 4 points)
A special case of Epstein-Zin-Weil preferences is the case where the utility function representating aversion to intertemporal consumption change is logarithmic. The following investment problem is based on an agent exhibiting such preferences with a coefficient of Arrow-Pratt risk aversion RRA $= 1 - \alpha$. The investor's choice problem is

$$\max_{c_1, \boldsymbol{\omega}} \log(c_1) + \frac{\beta}{\alpha} \log\left(\mathbb{E}\, c_2^{\alpha}\right)$$
$$\text{subject to } c_2 = [W - c_1]\boldsymbol{\omega}^{\intercal}\tilde{\boldsymbol{R}}$$
$$\text{and } \textstyle\sum_{i=1}^{n} \omega_i = 1\ .$$

where $W$ is his/her wealth, $\tilde{\boldsymbol{R}}$ is a vector of stochastic return factors and $\boldsymbol{\omega}$ the vector comprising the portfolio shares.

i) Derive the optimal level of first period consumption.

**Solution:** Substituting-in the first constraint and adding the other with a Lagrange multiplier delivers the maximization problem

$$\begin{aligned}
&\max_{c_1, \boldsymbol{\omega}} \log(c_1) + \frac{\beta}{\alpha} \log\left(\mathbb{E}\left([W - c_1]\boldsymbol{\omega}^{\intercal}\tilde{\boldsymbol{R}}\right)^{\alpha}\right) + \lambda(1 - \textstyle\sum_{i=1}^{n} \omega_i)\\
\Leftrightarrow &\max_{c_1, \boldsymbol{\omega}} \log(c_1) + \frac{\beta}{\alpha} \log[W - c_1]^{\alpha} + \frac{\beta}{\alpha} \log\left(\mathbb{E}\left(\boldsymbol{\omega}^{\intercal}\tilde{\boldsymbol{R}}\right)^{\alpha}\right) + \lambda(1 - \textstyle\sum_{i=1}^{n} \omega_i)\\
\Leftrightarrow &\max_{c_1, \boldsymbol{\omega}} \log(c_1) + \beta \log[W - c_1] + \frac{\beta}{\alpha} \log\left(\mathbb{E}\left(\boldsymbol{\omega}^{\intercal}\tilde{\boldsymbol{R}}\right)^{\alpha}\right) + \lambda(1 - \textstyle\sum_{i=1}^{n} \omega_i)
\end{aligned}$$

Optimizing first period consumption delivers the first order conditions

$$\begin{aligned}
&\frac{1}{c_1} + \frac{\beta}{W - c_1} = 0\\
\Rightarrow\quad &W - c_1 = \beta c_1\\
\Rightarrow\quad &c_1 = \frac{W}{1 + \beta}
\end{aligned}$$

ii) How does optimal consumption depend on the return factor? Please explain this finding using insights from our class.

**Solution:** Note: This is one is about the explanation part, which requires them to relate a problem set on income versus substitution effect under certainty to the type of stochastic portfolio investment problem with EZW preferences presented here. The first period consumption is independent of the return. The model disentangles risk aversion from intertemporal substitution. The intertemporal substitution is governed by log-utility. At the beginning of the class we learned that under log-utility the income and the substitution effect cancel each other and current consumption does not depend on the return factor.

**4. Intertemporal Optimization** (12 points, each question 4 points)
Assume a decision maker optimizes consumption given the following objective and constraints

$$\sum_{t=0}^{\infty} \beta^t u(c_t, B_t)$$

$$\begin{aligned} \text{subject to } \; K_{t+1} &= F(c_t, K_t, D_t) \\ B_{t+1} &= \alpha K_t \\ D_{t+1} &= D_t + f(B_t). \end{aligned}$$

i) Write down the corresponding Bellman equation. Interpret the value function and give an intuition for Bellman's dynamic programming equation (or Bellman's principle of optimality).

**Solution:** Let the value function be denoted $V(K_t, B_t, D_t)$. Note, getting the arguments of the value function right is the first part of the solution. Then, the Bellman equation is

$$V(K_t, B_t, D_t) = \max_{c_t} u(c_t, B_t) + \beta V(K_{t+1}, B_{t+1}, D_{t+1})$$

Interpretation of the value function:
It is the present value of stream of future utility that can be maximally derived in the model following an optimal trajectory of the control.
Intuition of Bellman's principle of optimality:
The maximal value that we can achieve over the infinite future is the max of the utility that we can achieve today and from tomorrow on (where the value from tomorrow on is discounted to today's value using $\beta$.)

ii) Assume you were given a trial-solution. How would you verify whether the trial solution is correct? Write down the main steps.

**Solution:** Steps:

1. Substitute-in the form of the trial solution in for $V(\cdot)$.

2. Substitute-in the equations of motion for the next period states.
3. Solve the maximization problem on the r.h.s. of the Bellman equation (FOCs).
4. Plug the solution of the optimization problem back into the r.h.s. of the Bellman equation (really only needed for cases where they depend on the state, otherwise sufficient to realize that optimal solution might not depend on the states).
5. Test whether the Bellman equation can be satisfied.
   Note: For this purpose you would usually collect all terms proportional to a given order of the states (or state-combinations) and see whether you can set of the corresposponding coefficients equal to zero, as well as satisfy the remaining state-independent part of the Bellman equation.

iii) How would the Bellman equation change in case the last dynamic equation changes from

$$D_{t+1} = D_t + f(B_t)$$

to the form

$$D_{t+1} = D_t + f(B_t, B_{t-1})?$$

**Solution:** The utility we can achieve over the infinite future from period $t$ on now also depends on the previous period's level of $B$, so the value function becomes $V(K_t, B_t, B_{t-1}, D_t)$ and the Bellman equation accordingly

$$V(K_t, B_t, B_{t-1}, D_t) = \max_{c_t} u(c_t, B_t) + \beta V(K_{t+1}, B_{t+1}, B_t, D_{t+1})$$