Economic Dynamics and Uncertainty
Final Exam – Fall 2020

Problems 2 and 3 are brief discussions that you can type into INSPERA. If you are concise a paragraph for each question will be enough. Problem 1 requires calculations and can be answered on paper. It is your responsibility that the submitted document/photo/scan is legible. If you get stuck or think that a question is ambiguous explain your concern and pick your interpretation or make an explicit assumption if needed, and move on.

1. Optimal consumption with fixed income (50%).
Consider a consumer with an initial wealth W_0 who receives a fixed income F at the beginning of each period. The interest rate is r and there is a perfect credit market so the consumer can borrow on future income as long as he is able to pay back in the final period. We will assume that $r = 0$. The consumer choses consumption C_t each period and maximizes

$$\sum_{t=0}^{T} \beta^t \ln(C_t)$$

subject to the wealth accumulation equation

$$W_{t+1} = (W_t - C_t) + F$$

(Remember that $r = 0$.) Let $V_T(W)$ denote the value function when there are T periods left (in addition to the current, i.e., here $W = W_0$).

i) Solve the case $T = 1$, i.e., find the optimal C_0 and the value function $V_1(W_0)$.

> **Solution:** Note that this is like a cake eating economy with wealth $W + F$. Optimal consumption is $\frac{W+F}{1+\beta}$.

ii) Solve the problem for $T = 2$, i.e., find optimal C_0 and the value function $V_2(W)$

> **Solution:** *The simplest solution is to recognize that this is the cake eating economy with wealth $W'' = W + 2F$, Hence we can use the results about the cake eating economy:*
>
> $$C = \frac{W + 2F}{1 + \beta + \beta^2}$$
>
> *Or, using a) we get*
>
> $$V_2(W) = (1 + \beta)\ln(W + F) + B$$
>
> *Where B is a constant. Thus the maximizing the RHS of the Bellman equation becomes (leaving out constant terms):*
>
> $$\max \ln C + \beta(1 + \beta)\ln(W + F - C + F)$$

withthe FOC

$$\frac{1}{C} = \frac{\beta(1+\beta)}{W+2F-C}$$

with solution

$$C = \frac{W+2F}{1+\beta+\beta^2}$$

iii) Since the utility function is $\ln C$, we may guess that the value function in the infinite horizon case ($T = \infty$) is of the form

$$V(W) = A \ln W + B$$

Check whether this guess is correct.

Solution: *It is not, we solve*

$$\begin{aligned} V(W) &= \max_C \ln(C) + \beta V(W - C + F) \\ &= \max_C \ln(C) + \beta A \ln(W - C + F) + B \end{aligned}$$

FOC

$$\begin{aligned} \frac{1}{C} &= \frac{\beta A}{W - C + F} \\ (1+\beta A)\, C &= W + F \\ C &= \frac{W+F}{1+\beta A} \\ W - C + F &= \beta A \frac{W+F}{1+\beta A} \end{aligned}$$

Hence

$$V(W) = (1+\beta A)\ln(W+F) + B'$$

But as the function $\ln(W+F)$ *cannot be transformed into a* $\ln(W)$, *we conclude that* $V(W) = A \ln W + B$ *is an incorrect guess.*

iv) Based the results about the cake eating economy in the textbook and on a) and b) above, explain why we may expect that V_T is of the form

$$V_T(W_0) = \left(\sum_{t=0}^{T-1} \beta^t\right) \ln(W_0 + TF) + B$$

Solution:

This is the value function for the T-period cake eating economy with initial wealth $W_0 + TF$ *which is exactly what the consumer can spend over T periods here.*

v) Given the value function above, what happens to initial consumption C_0 in the T period problem, as $T \to \infty$.

Solution: *Using the value function stated above and maximizing the RHS of the Bellman equation gives* $C_{0,T+1}$

$$\max \ln(C_0) - \beta \left(\sum_{t=0}^{T-1} \beta^t \right) \ln(W_0 + TF - C + T)$$

$$\frac{1}{C} = \beta \left(\sum_{t=0}^{T-1} \beta^t \right) \frac{1}{W_0 + (T+1)F - C}$$

$$C_{0,T+1} = \frac{W_0 + (T+1)F}{\sum_{t=0}^{T} \beta^t} \to \infty \text{ as } T \to \infty$$

2. Numerical solution (30%).

If we did not know the analytical solution to Problem 1 above, we could try solve the problem numerically using Chebychev polynomials. We then only consider the case $T < \infty$.

i) Explain briefly how you would solve the problem numerically.

Solution: Note that we know $V_0(W)$. We can compute $V_1(W)$ on a set of chebychev nodes maximizing the right hand side of the Bellman equation, and then fit a Chebychev polynomial thorugh the computed function values at these node. This will be our approximation of $V_1(W)$. Having an estimate of $V_1(W)$ we can compute $V_2(W)$ on the set of chebychev nodes in the same manner and compute the Chebychev polynomials approximating $V_3(W)$. We continue this iteration until we reach V_T. To check that the order of the polynomial is sufficient for an accurate answer we can repeat the process with higher order polynomial until further increases does not change the answer significantly.

ii) Is there any way you could evaluate the accuracy of the numerical solution?

Solution: The relevant numeric inaccuracies arise from having to pick a finite basis and from (when) using a finite approximation interval. As a result the Chebychev polynomials only approximate the true value function. Adding higher order polynomials can increase the accuracy. We can start out with Chebychev polynomials of a certain degree. If the results is reasonably accurate it should not change if we add higher order polynomials nor if we extend the approximation interval. Another possible test is to evaluate the final iteration on a finer grid of evaluation points and check the error, i.e., the difference between the values calculated from the r.h.s. value function and merely evaluating the period zero value function. We do not expect the latter answer, but only some reasonable discussion of how to evaluate numerical accuracy.

3. Asset Pricing (20%).

Please comment on the statement: "A high covariance of an asset's return with consumption explains a high risk premium". In particular, explain the intuition why this statement might (or might not) hold.

Solution: The main expected answer relates to the consumption-based asset pricing model. Here, a high covariance of consumption with an asset's return suggests that the asset pays highly in times where consumption is high and, thus, marginal utility derived from more consumption is low. Thus, in order to invest in such an asset, it has to have a higher return. A more elaborate answer might also comment on the fact that the consumption based asset pricing model does not actually perform that well and, thus, the statement is empirically not as convincing, but we this part was not required.