Economic Dynamics and Uncertainty
Final Exam – Fall 2020

Problems 1 and 2 are brief discussions that you can type into INSPERA. If you are consise, half a page should be enough. Please try not to exceed one page. Problem 3 requires calculations and can be answered on paper. It is your responsibility that the submitted document/photo/scan is legible.

1. Income and Substitution Effect (20%).
Write a brief discussion of the intertemporal income and substitution effect. Include in your discussion for each of these effects and a discussion of which of the two dominates and what it implies for present consumption. Then, please briefly discuss the effect of risk aversion and uncertain returns when the intertemporal income and sustitution effect balance each other.

Solution: Basic answer:

We can split up the current consumption response to a change in interest into the intertemporal income and substitution effects. (i) For an intuition, let us examine the case of an increase in the interest rate. Then, the intertemporal income effect captures that an increase in interest increases overall life-time income and, thus, makes the agent consume more in the present as he or she has more life-time income to spend. The interetemporal substitution effect captures that the increase in interest makes it more valuable to save today (consume less) and instead consume tomorrow where the consumer now gets relatively "more bang for the buck". Thus, the substitution effect tends to reduce present consumption if the interest increases. (ii) Which of the two effects is stronger depends on the consumer's desire to smooth consumption over time. If the consumer has strong desire to smooth consumption over time (precisely, $RRA > 1$ or an elasticity of intertemporal substitution smaller than unity), he or she will not substute much consumption from the present to the future and the income effect will dominate, increasing current consumption in response to an increase in interest. If the consumer has a low desire to smooth consumption over time ($RRA < 1$), then the substitution effect will dominate and consumption in the present falls in response to an increase in interest. (iii) This one was a bit more open ended. An answer I would like particularly is relating this point to problem set 8 where we saw that if intertemporal income and substitution effect cancel each other because the desire to smooth consumption of time (or the elasticity of inertemporal substitution) is unity, then (disentangled) Arrow-Pratt risk aversion and uncertainty about the return have no impact on present consumption.

2. Numeric Dynamic Programming (20%).
Write a brief(!) discussion how you would solve a stationary infinite horizon optimization problem numerically on your computer using dynamic programming. Include in your discussion (i) what the main challenge is in solving the Bellman equation rather than a simple two period optimization problem, (ii) how you would represent the value function, (iii) how

you would update the value function over the course of your iterations, (iv) when you would consider the Bellman equation solved. We are fine with solving the Bellman equation and you do not have to explain how to generate a time path.

Solution: The main challenge in solving the Bellman equation is that we do not generally know the form of the value function. The optimization problem in the Bellman equation looks like a two period optimization problem, but the fact that we do not know the value function turns it into a functional equation.

To solve the problem, we approximate the value function, e.g., by Chebychev polynomials over the desired solution interval (alternativley we can use splines or neural networkds or...). After deciding on the interval, we pick a number of polynamials (basis function) that we want to use to approximate the true value function. Depending on this choice, we generate the optimal approximation points, e.g., Chebychev nodes. In each iteration, we evaluate the right hand side Bellman equation on these evaluation nodes. For this purposes, we solve the r.h.s. Bellman equation, including the two period optimization problem, given some guess of the value function for each of our grid points. Once we have evaluated the right hand side of the Bellman equation, we use these values on our grid points to calculate a better estimate of the coefficients approximating our value function using the Chebychev basis. We repeat this iteration until the basis coefficients start changing by very little (e.g. 10^{-3}). Alterantivley, we stop the iteration when the absolute difference of the value function between two iterations becomes very little (preferably in relative terms - but careful in case values are close to zero). Then we consider the value function problem solved.[1]

3. Trading-off consumption and leisure (60%).

A representative agent derives utility from consumption C_t and leisure L_t in the form

$$U(C_t, L_t, \bar{C}_t) = \log\left(C_t^{1-\alpha-\gamma}\ L_t^{\gamma}\left(\frac{C_t}{\bar{C}_t}\right)^{\alpha}\right).$$

The consumption level $\bar{C}_t$ reflects the average consumption level in the economy. She discounts the future with discount factor β and has an infinite planning horizon. The economy's production follows a Cobb-Douglas composition of labor $\tilde{L}_t$, capital K_t, and the exogenously evolving technology level A_t yielding output

$$Y_t = A_t\ K_t^{\kappa}\ \tilde{L}_t^{1-\kappa}.$$

A time constraint limits labor $\tilde{L}_t$ and leisure L_t to the available hours N_t, i.e., $\tilde{L}_t + L_t = N_t$. The economy's dynamic equation governs the capital stock

$$K_{t+1} = Y_t - C_t, \tag{1}$$

which assumes full depreciation of capital in the production process. Assume an interior solution to all optimization problems.

i) A representative agent maximizes utility over consumption and leisure subject to the production function taking the average consumption level of the economy $\bar{C}_t$ as given. She uses the production function and evoluation of capital in the economy as her income constraint.[2]

Assume that a linear-affine value function (linear in log-capital and with an additive constant) solves the dynamic programming problem and write down the corresponding Bellman equation.

Note: It is not necessary but can be helpful to express the consumption control in terms of the consumption rate. Then, you should show that the (log) capital equation is $k_{t+1} = a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t)$, where we use lower case letters for the log of a variable and x_t for the consumption rate.

Solution: Expressing utility

$$U(C_t, L_t, \bar{C}_t) = (1-\alpha-\gamma)\log(C_t) + \gamma\log(L_t) + \alpha\log\left(\frac{C_t}{\bar{C}_t}\right)$$
$$= (1-\gamma)\log(C_t) + \gamma\log(L_t) - \alpha\log\left(\bar{C}_t\right)$$

in terms of the consumption rate $x_t = \frac{C_t}{Y_t}$ we find

$$u(x_t, L_t, \bar{C}_t) = (1-\gamma)\left(\log(x_t) + \log(Y_t)\right) + \gamma\log(L_t) - \alpha\log\left(\bar{C}_t\right) =$$
$$(1-\gamma)\Big(\log(x_t) + \log(A_t) + \kappa\log(K_t) + (1-\kappa)\log(\tilde{L}_t)\Big) + \gamma\log(L_t) - \alpha\log\left(\bar{C}_t\right)$$

where $\tilde{L}_t = N_t - L_t$. Using lower case letters for the log of a variable we have

$$u(x_t, l_t, \bar{c}_t) = (1-\gamma)\Big(\log(x_t) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t\Big) + \gamma l_t - \alpha\bar{c}_t$$

where $\tilde{l}_t = \log(N_t - \exp(l_t))$.
Similarly we transform the equation of motion from

$$K_{t+1} = Y_t - C_t = Y_t(1-x_t) = A_t\, K_t^{\kappa}\, \tilde{L}_t^{1-\kappa}(1-x_t) \tag{2}$$

to its log form

$$k_{t+1} = a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t).$$

Note: The transformation of consumption into a consumption rate is helpful but not necessary, the transformation of the equation of motion into log-capital is necessary to solve the Bellman equation for our trial solution.

[2] This unusual constraint for a representative agent makes your life a lot easier and permits a simpler comparison with the subsequent social planner problem.

The linear-affine value function is of the form $V(K_t, t) = \varphi k_t + \varphi_t$ delivering the Bellman equation

$$
\begin{aligned}
V(k_t, t) \quad &= \max_{x_t, l_t} u(x_t, l_t, \bar{c}_t) + \beta V(k_{t+1}, t+1) \\
\Rightarrow \varphi k_t + \varphi_t &= \max_{x_t, l_t} (1-\gamma)\Big(\log(x_t) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t \Big) + \gamma l_t - \alpha \bar{c}_t \\
&\quad + \beta\varphi \Big(a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t) \Big) + \beta \varphi_{t+1}
\end{aligned}
$$

ii) Solve the first order condition for leisure.

Solution:

Note: As in the examples in class, our transformation into consumption rates implies that the r.h.s. Bellman equation separates the controls, here x_t and l_t, from the states, here k_t. As a result we can carry out the optimization independent of the value of k_t giving us the optimal x_t^* and l_t^*.

Taking the first order condition with respect to leisure and keeping in mind that $l_t = \log(L_t)$ and $\tilde{l}_t = \log(N_t - L_t)$ we find

$$
\begin{aligned}
&\max_{L_t} (1-\gamma)\Big(\log(x_t) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t \Big) + \gamma l_t - \alpha \bar{c}_t \\
&\quad + \beta\varphi \Big(a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t) \Big) + \beta \varphi_{t+1} \\
&\Rightarrow (1-\gamma)(1-\kappa)\frac{d\tilde{l}_t}{dL_t} + \gamma \frac{dl_t}{dL_t} + \beta\varphi(1-\kappa)\frac{d\tilde{l}_t}{dL_t} = 0 \\
&\Rightarrow \gamma \frac{1}{L_t} = \big(\beta\varphi(1-\kappa) + (1-\gamma)(1-\kappa) \big) \frac{1}{N_t - L_t} \\
&\Rightarrow \gamma N_t = \gamma L_t + \big(\beta\varphi(1-\kappa) + (1-\gamma)(1-\kappa) \big) L_t \\
&\Rightarrow L_t = \frac{\gamma N_t}{\gamma + (1-\gamma+\beta\varphi)(1-\kappa)}
\end{aligned}
$$

iii) Solve the first order condition for consumption or for the consumption rate.

Solution:

Taking the first order condition with respect to the consumption rate (a strictly monotonic transformation of consumption) yields

$$\max_{x_t} (1-\gamma)\Big(\log(x_t) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t\Big) + \gamma l_t - \alpha \bar{c}_t$$
$$+\beta\varphi\left(a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t)\right) + \beta\varphi_{t+1}$$
$$\Rightarrow (1-\gamma)\frac{1}{x_t} + \beta\varphi\left(-\frac{1}{1-x_t}\right) = 0$$
$$\Rightarrow 1-\gamma-(1-\gamma)x_t = \beta\varphi x_t$$
$$\Rightarrow x_t = \frac{1-\gamma}{1-\gamma+\beta\varphi}$$

iv) Now, assume that a social planner maximizes utility over consumption C_t and leisure L_t subject to the production function, incorporating that $\bar{C}_t = C_t$. You may again pick the consumption rate as the control variable rather than aggregate consumption itself.

Assume again that a linear-affine value function in log-capital solves the dynamic programming problem. Derive the optimal leisure and consumption choices from the perspective of the social planner.

Solution: Now C_t and $\bar{C}_t$ cancel in the final utility term

$$U(C_t, L_t, \bar{C}_t) = (1-\alpha-\gamma)\log(C_t) + \gamma\log(L_t) + \alpha\log(1)$$
$$= (1-\alpha-\gamma)\log(C_t) + \gamma\log(L_t)$$

changing the two blue terms in the Bellman equation delivering instead

$$\varphi k_t + \varphi_t = \max_{x_t, l_t} (1-\alpha-\gamma)\Big(\log(x_t) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t\Big) + \gamma l_t$$
$$+\beta\varphi\left(a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t)\right) + \beta\varphi_{t+1}.$$

As a result of the additional "$-\alpha$" the solution for leisure changes to

$$\max_{L_t} (1-\alpha-\gamma)\Big(\log(x_t) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t\Big) + \gamma l_t$$
$$+\beta\varphi\left(a_t + \kappa k_t + (1-\kappa)\tilde{l}_t + \log(1-x_t)\right) + \beta\varphi_{t+1}$$
$$\Rightarrow L_t = \frac{\gamma N_t}{\gamma + (1-\alpha-\gamma+\beta\varphi)(1-\kappa)},$$

and the FOC for the consumption rate changes to

$$\max_{x_t} (1-\alpha-\gamma)\Big(\log(x_t)+a_t+\kappa k_t+(1-\kappa)\tilde{l}_t\Big)+\gamma l_t$$
$$+\beta\varphi\left(a_t+\kappa k_t+(1-\kappa)\tilde{l}_t+\log(1-x_t)\right)+\beta\varphi_{t+1}$$
$$\Rightarrow x_t=\frac{1-\alpha-\gamma}{1-\alpha-\gamma+\beta\varphi}.$$

v) Do the expressions for the optimal controls differ or coincide? At this point, can you meaningfullly compare the consumption rates across the two different scenarios? Discuss.

Solution: Conditionally on the shadow value of log-capital, the expressions differ. Yet, we can compare the two solutions only conditionally on the shadow value of capital. In general, both problems result in different shadow values. Moreover, the two utility functions differ and, thus, the meaning of the shadow value in utils also differs. That makes it hard to draw any conclusions from the differences in

$$x_t^{priv}=\frac{1-\gamma}{1-\gamma+\beta\varphi}$$
$$x_t^{social}=\frac{1-\alpha-\gamma}{1-\alpha-\gamma+\beta\varphi}.$$

vi) Solve for the shadow value of log-capital in each of the two problems. You can treat them jointly. Do the shadow values differ?

Solution: To treat both problems simultaneously, let $\delta=\alpha$ for the representative agent case and $\delta=0$ for the social planner, and let $\eta=\gamma$ for the representative agent case and let $\eta=\alpha+\gamma$ for the social planner. Then the Bellman equation (in either case) is

$$\varphi k_t+\varphi_t=\max_{x_t,l_t}(1-\eta)\Big(\log(x_t)+a_t+\kappa k_t+(1-\kappa)\tilde{l}_t\Big)+\gamma l_t-\delta\bar{c}_t \text{ and}$$
$$+\beta\varphi\left(a_t+\kappa k_t+(1-\kappa)\tilde{l}_t+\log(1-x_t)\right)+\beta\varphi_{t+1}.$$

Plugging in the optimal controls (which generally differ for the two cases), denoting them by stars, we find

$$\varphi k_t + \varphi_t = (1-\eta)\Big(\log(x_t^*) + a_t + \kappa k_t + (1-\kappa)\tilde{l}_t^*\Big) + \gamma l_t^* - \delta \bar{c}_t$$
$$+\beta\varphi\left(a_t + \kappa k_t + (1-\kappa)\tilde{l}_t^* + \log(1-x_t^*)\right) + \beta\varphi_{t+1}.$$

To solve the dynamic programming equation, the Bellman equation has to be satisfied for all relevant realizations of the state k_t. A necessary condition for such a solution is that the terms proportional to k_t cancel each other, implying

$$\begin{aligned} &\varphi - (1-\eta)\kappa - \beta\varphi\kappa = 0 \\ \Rightarrow\ &\varphi(1-\beta\kappa) = (1-\eta)\kappa \\ \Rightarrow\ &\varphi = \frac{(1-\eta)\kappa}{1-\beta\kappa} \end{aligned}$$

where $\eta = \gamma$ in the case of the representative agent and $\eta = \alpha + \gamma$ in the case of the social planner. So yes, they differ.

vii) Does the linear-affine value function indeed solve the Bellman equation? Explain.

Solution: Yes, it does. Given the shadow value for log-capital calculated in the previous excercise all terms in the Bellman equation depending on captial cancel each other. In addition, we can find a sequence of shadow values $\varphi_t, \varphi_{t+1}, \ldots$ such that also the affine terms, i.e., those independent of the state cancel each other across both sides of the Bellman equation. Then the Bellman equatin is satisfied for all realizations of the state.

viii) Using your result for the shadow value of capital, calculate the optimal leisure levels and the optimal consumption rate for the representative agent and for the social planner.

Solution: For the representative agent

$$\varphi = \frac{(1-\gamma)\kappa}{1-\beta\kappa}$$

delivers

$$L_t = \frac{\gamma N_t}{\gamma + (1-\gamma+\beta\varphi)(1-\kappa)} = \frac{\gamma N_t}{\gamma + (1-\gamma+\beta\frac{(1-\gamma)\kappa}{1-\beta\kappa})(1-\kappa)}$$

$$= \frac{(1-\beta\kappa)\gamma N_t}{\gamma - \gamma\beta\kappa + \Big((1-\gamma)(1-\beta\kappa) + \beta\kappa - \gamma\beta\kappa\Big)(1-\kappa)}$$

$$= \frac{(1-\beta\kappa)\gamma N_t}{\gamma - \gamma\beta\kappa + (1-\gamma)(1-\kappa)} = \frac{(1-\beta\kappa)\gamma N_t}{-\gamma\beta\kappa + 1 - \kappa + \gamma\kappa} = \frac{(1-\beta\kappa)\gamma N_t}{1-(1-\gamma+\beta\gamma)\kappa}$$

and

$$x_t = \frac{1-\gamma}{1-\gamma+\beta\varphi} = \frac{1-\gamma}{1-\gamma+\beta\frac{(1-\gamma)\kappa}{1-\beta\kappa}} = \frac{1}{1+\frac{\beta\kappa}{1-\beta\kappa}} = 1-\beta\kappa.$$

For the social planner

$$\varphi = \frac{(1-\alpha-\gamma)\kappa}{1-\beta\kappa}$$

delivers

$$L_t = \frac{\gamma N_t}{\gamma + (1-\alpha-\gamma+\beta\varphi)(1-\kappa)}$$

$$= \frac{\gamma N_t}{\gamma + (1-\alpha-\gamma+\beta\frac{(1-\alpha-\gamma)\kappa}{1-\beta\kappa})(1-\kappa)}$$

$$= \frac{(1-\beta\kappa)\gamma N_t}{\gamma - \gamma\beta\kappa + \Big((1-\beta\kappa)(1-\alpha-\gamma) + (1-\alpha-\gamma)\beta\kappa\Big)(1-\kappa)}$$

$$= \frac{(1-\beta\kappa)\gamma N_t}{\gamma - \gamma\beta\kappa + (1-\alpha-\gamma)(1-\kappa)}$$

$$= \frac{(1-\beta\kappa)\gamma N_t}{-\gamma\beta\kappa + 1 - \alpha - \kappa + \kappa\alpha + \kappa\gamma)}$$

$$= \frac{(1-\beta\kappa)\gamma N_t}{1-(1-\gamma+\gamma\beta)\kappa - \alpha(1-\kappa)}$$

and for the consumption rate

$$x_t = \frac{1-\alpha-\gamma}{1-\alpha-\gamma+\beta\varphi}$$

$$= \frac{1-\alpha-\gamma}{1-\alpha-\gamma+\beta\frac{(1-\alpha-\gamma)\kappa}{1-\beta\kappa}} = 1-\beta\kappa.$$

ix) Interpret possible differences you find in leisure and the consumption rate for the representative agent's solution versus the social planner's solution. Discuss your intuition why they differ.

Solution: Leisure is higher in the social planner regime, the consumption rate is the same. The representative agent evaluates her welfare relative to the average consumption level (with weight α). Thus, she has an additional incentive to be better off than the average. To this end, she works harder. However, on the social level, if everyone works harder, this attempt of trying to increase consumption relative to the average is futile. The social planner incorporates that everyone trying harder to be better than the average only increases the average consumption level. None of the individuals will end up being better than the average (given we assume a representative agent and we don't distinguish different types). The social planner's objective therefore turns of the futile attempt to "keep up with the Joneses". Instead, the social planner solution suggest that everyone should work a little less and, yet, in relative terms, everyone remains equally well off.

x) Discuss how absolute consumption, GDP, and capital growth differ between the two settings.

Solution: In the social planner regime, leisure is higher. Thus, GDP is lower in the social planner regime. Given the consumption rates coincide, consumption is lower and so is capital accumulation.

If you think that your result and interpretation suggest that you made calculus or other mistakes along the way, you are invited to discuss your suspicion and what you would have expected instead.