Economic Dynamics and Uncertainty
Final Exam – Fall 2023

Each roman numbered subquestion counts equally to the exam. The final starred subquestion 2.vi*) is a bonus question.

1. Asset Pricing (40%).

You are advising an educated economist who is ignorant about asset pricing on how to value a given asset. She is convinced that the correlations with the return on $\tilde{x}$ should be the crucial factor in determining the asset's return and heard that there is a model of the form

$$\mathbb{E}\,\tilde{R}_i = \alpha + \beta(\tilde{R}_i, \tilde{R}_x)\eta$$

that she can use to price the asset, but is not quite sure about the meaning of this equation.

i) Please explain the different terms in the equation.

> **Solution:** The equations are different ways of writing the generic one factor model with
>
> - pricing factor $\tilde{x}$ with return $\tilde{R}_x$
> - $\tilde{R}_i$ is an asset i's stochastic return, $\mathbb{E}\,\tilde{R}_i$ its expected return
> - $\beta(\tilde{R}_i, \tilde{R}_x)$ is the regression coefficient of the asset's return on the factor $\tilde{x}$:
>
> $$\beta(\tilde{R}_i, \tilde{x}) = \frac{\mathrm{Cov}(\tilde{R}_i, \tilde{R}_x)}{\mathrm{Var}(\tilde{R}_x)}.$$
>
> - η is (the factor) *risk premium.*
> - α is called the *zero beta return* (it is the return if *beta*$(\cdot)$ is zero). Saying it's a constant is OK.

ii) Explain how she can get from the equation above to the alternative form

$$\mathbb{E}\,\tilde{R}_i - R_f = \beta(\tilde{R}_i, \tilde{R}_x)\left(\mathbb{E}\,\tilde{R}_x - R_f\right).$$

Explain steps and possible assumptions and relate the terms back to the earlier equation.

> **Solution:** To get from the first to the second equation:

1. Observe that $\beta(\tilde{R}_x, \tilde{R}_x) = 1$ and

$$\mathbb{E}\,\tilde{R}_x = \alpha + \beta(\tilde{R}_x, \tilde{R}_x)\eta = \alpha + \eta \quad \Rightarrow \quad \eta = \mathbb{E}\,\tilde{R}_x - \alpha$$

and therefore the pricing equation of asset i can be written as

$$\mathbb{E}\,\tilde{R}_i - \alpha = \beta(\tilde{R}_i, \tilde{R}_x)\left(\mathbb{E}\,\tilde{R}_x - \alpha\right).$$

2. The next step requires the assumption that there exists a risk-free asset. If such a risk-free asset exists, it follows that $R_f = \mathbb{E}\,\tilde{R}_f = \alpha$ because the covariance with a constant is zero, delivering the second equation.

In this equation, we have

- the assets expected excess return $\tilde{R}_i - R_f$ on the left
- and the factor risk premium η is now expressed as the excess return of the pricing factor itself

iii) The economist tells you that she had not seen the previous equation, but this one instead:

$$\bar{R}_i - R_f = \rho_{i,M}\ \sigma_i\ S_e$$

Explain the different terms in this equation, and state which factor pricing model this equation most likely corresponds to. Explain the insights deriving from this equation.

Solution:

Given that the excess return here relies on the correlation with a variable labeled M, we are most likely dealing with the CAPM (capital asset pricing model). [if consistently arguing with a different model that's fine - but likely harder]. As before we have the excess return of asset i on the left. The new terms are

- $\tilde{R}_M$ return to the market portfolio
- $\rho_{i,M}$ is the correlation coefficient between the asset return and the return of the market portfolio $\left(\rho_{i,M} = \frac{\mathrm{Cov}(\tilde{R}_i, \tilde{R}_M)}{\sigma_i \sigma_m}\right)$
- σ_i standard deviation of asset return
- market portfolio's Sharpe ratio

$$S_e \equiv \frac{\bar{R}_M - R_f}{\sigma_M}.$$

It measures the market portfolio's excess return per unit of risk.

The equation tells us that, in this model, an asset's excess return depends on it's correlation with the market portfolio's return. In particular, it states that only the part of an asset's volatility (described by σ_i) correlated with the market portfolio is relevant to price the asset. The priced risk $\sigma_i \rho_{i,M}$ is priced according to the Sharpe ratio, which measures the excess return of the market portfolio per unit of risk.

Note1: More explicitly the equation above can be written as

$$\bar{R}_i - R_f = \frac{\text{Cov}(\tilde{R}_i, \tilde{R}_M)}{\sigma_M} \frac{\bar{R}_M - R_f}{\sigma_M} = \rho_{i,M}\ \sigma_i\ \frac{\bar{R}_M - R_f}{\sigma_M} = \rho_{i,M}\ \sigma_i\ S_e$$

Note 2: The term $\rho_{i,M}\ \sigma_i$ captures the marginal increase in market risk σ_M resulting from a marginal increase of asset i in the market portfolio.

iv) The economists tells you that she's not convinced that the pricing factor of the previous model is really the only determinant for asset returns. She asks you whether there might be models that use more than one factor. Give a briefly(!) verbal answer (no formulas needed). If your answer is affirmative, motivate why there might be more than one factor in such a model. If it is negative, explain why not.

Solution: Yes, there are models that use more than one pricing factor, e.g., the Fama-French model uses three pricing factors. Apart from the market portfolio return they also use the pricing factors "**small-minus-big (SMB)**" characterizing the return on a portfolio of stocks from small companies minus the return on a portfolio of stocks from large companies and "**high-minus-low (HML)**" characterizing the return on a portfolio of stocks issued by firms with a high book-to-market value ("value stocks") minus the return on a portfolio of stocks in firms with a low book-to-market value ("growth stocks").

Static consumption-based analysis generally leads to a single pricing factor. One motivation to use more than one pricing factors is that the relation between the stochastic discount factor $\tilde{M}_t$ and our observables might change over time. If we cannot perfectly condition on all the changes, then the unconditional model will typically depend on more than one pricing factor, where the additional pricing factors represent the time changing relations.

Alternatively, one could argue that Epstein-Zin preferences give rise to a second pricing factor.

2. Dynamic Programming (60%). A social planner with infinite planning horizon and utility discount factor β maximizes a representative agent's utility $U(c_t, x_t)$, where x_t is the level of biodiversity and consumption c_t increases utility but also destroys some of the biodiversity. Utility is increasing and concave in both arguments. The law of motion for biodiversity is

$$x_{t+1} = f(x_t) - g(c_t)$$

with some given functions f and g.

Note: Biodiversity is a measure for species diversity. We reduce biodiversity by destroying species, e.g., through habitat destruction or climatic change.

i) Spell out the social planner's Bellman equation including all necessary substitutions and/or constraints required to solve the problem.

Solution: The Bellman or dynamic programmign equation is

$$v(x_t) = \max_{c_t} \left[U(c_t, x_t) + \beta v(f(x_t) - g(c_t))\right] \tag{1}$$

with x_0 given and $c_t, x_t \geq 0$.

ii) Derive the Euler equation from the dynamic programming equation.

Solution: Step 1: First order condition of r.h.s. Bellman

$$\frac{\partial U(c_t, x_t)}{\partial c_t} = \beta v'(x_{t+1}) g'(c_t) \tag{2}$$

or

$$v'(x_{t+1}) = \frac{\frac{\partial U(c_t,x_t)}{\partial c_t}}{\beta g'(c_t)} \tag{3}$$

Step 2: Derive Bellman equation w.r.t. x_t and make use of the Envelope theorem

$$v'(x_t) = \frac{\partial U(c_t, x_t)}{\partial x_t} + \beta v'(x_{t+1}) f'(c_t)$$

It's easiest to advance the equation by one period

$$v'(x_{t+1}) = \frac{\partial U(c_{t+1}, x_{t+1})}{\partial x_{t+1}} + \beta v'(x_{t+2}) f'(x_{t+1}). \tag{4}$$

Step 3: We can use equation (3) in t and $t+1$ to substitute the unknown value function in the previous equation

$$\frac{\frac{\partial U(c_t,x_t)}{\partial c_t}}{\beta g'(c_t)} = \frac{\partial U(c_{t+1}, x_{t+1})}{\partial x_{t+1}} + \beta \frac{\frac{\partial U(c_{t+1},x_{t+1})}{\partial c_{t+1}}}{\beta g'(c_{t+1})} f'(x_{t+1}) \tag{5}$$

We successfully eliminated the unknown value function.
Step 4: Rearranging yields

$$\begin{aligned}
\frac{\frac{\partial U(c_t,x_t)}{\partial c_t}}{\beta g'(c_t)} &= \frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}} + \beta \frac{\frac{\partial U(c_{t+1},x_{t+1})}{\partial c_{t+1}}}{\beta g'(c_{t+1})} f'(x_{t+1}) \\
\Leftrightarrow \frac{\partial U(c_t,x_t)}{\partial c_t} &= \beta \frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}} g'(c_t) + \beta \frac{g'(c_t)}{g'(c_{t+1})} \frac{\partial U(c_{t+1},x_{t+1})}{\partial c_{t+1}} f'(x_{t+1}) \\
\Leftrightarrow \frac{\partial U(c_t,x_t)}{\partial c_t} &= \beta \left[\frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}} + \frac{f'(x_{t+1})}{g'(c_{t+1})} \frac{\partial U(c_{t+1},x_{t+1})}{\partial c_{t+1}} \right] g'(c_t).
\end{aligned}$$

which is the general Euler equation.

iii) Interpret the Euler equation.
If you did not succeed in deriving the Euler equation, assume the Euler equation has a term of the form $\beta \frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}} g'(c_t)$ on the right hand side and explain why this novel term appears in the Euler equation and interpret it.

Solution:

$$\frac{\partial U(c_t,x_t)}{\partial c_t} = \beta \left[\frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}} + \frac{f'(x_{t+1})}{g'(c_{t+1})} \frac{\partial U(c_{t+1},x_{t+1})}{\partial c_{t+1}} \right] g'(c_t).$$

The left of the Euler equation states the marginal utility that the representative agents derives from a unit of consumption today. The right hand side states the marginal discounted value that he or she can derive in the next period from giving up a unit of consumption today. Giving up this marginal unit today reduces tomorrow's loss in biodiversity by $g'(c_t)$ units. The additional biodiversity implies the immediate additional biodiversity utility stream $\frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}}$. Moreover, given the additional biodiversity level at the beginning of the next period, the agent can consume more in the second period while leaving behind the same biodiversity stock for the future (as compared to the scenario without the marginal consumption reduction in the present). The precise ratio of how much the agent can consume in addition while leaving behind the same biodiversity stock is determined by the ratio $\frac{f'(x_{t+1})}{g'(c_{t+1})}$.

The most weight in this answer will be given to the novel term $\frac{\partial U(c_{t+1},x_{t+1})}{\partial x_{t+1}} g'(c_t)$.

iv) Let's think about a generalization where the agent has two types of consumption. One type of consumption labeled c_2 relies on activities that destroys biodiversity, the other labeled c_1 does not. The utility derived from consumption is now $U(ac^s_{1,t} + (1-a)c^s_{2,t}, x_t)$. Let's assume that consumption level c_1 evolves exogenously and the equation of motion for biodiversity is now

$$x_{t+1} = f(x_t) - g(c_{2,t}).$$

How does this change in the model affect the Euler equation (intertemporal trade-off) previously derived?

Hint: Think, rather than re-solving the problem. You can can answer this question even if you have not solved (or solved correctly/fully) for the Euler equation above.

Solution: The only effective change is that marginal utility now changes exogenously over time along with $c_{2,t}$.

Note: We don't expect students to worry about the magnitude of s. If students comment on the fact that if $s > 1$ concavity of the utility might no longer be guaranteed or that for $s < 0$ the utility function, to be increasing in consumption, has to differ from the U above that gives additional credit.

v) What would be missing in our model to meaningfully let the representative agent optimize over the consumption level c_2.

Hint: The correct answer is related to questioning whether the original trade-off in the model with only c_t really contained the comprehensive trade-off we might want in a more realistic model.

Solution: The problem assumed that the only reason the agent did not consume arbitrary levels is because it destroys biodiversity. What is missing is some other production or scarcity constraint. In order to enable the model to optimize as well over c_2 (or to more meaningfully optimize over consumption in the previous subproblems), we would need either a static consumption constraint (thought that would be a bit boring in the case of c_2 as the agent would always consume at the constraint) or, more interestingly, we would need another equation of motion that capture production dynamics and the resulting consumption contraint (a trade-off between consumption and investment).

Bonus: vi*) Let's return to the original optimization problem posed in the first part of this problem. Do you think the equation captures well a model of biodiversity loss? Think about how easy or difficult it is to regrow biodiversity as compared to eliminating biodiversity. Discuss how you might want to modify the model to better capture biodiversity loss. No need to actually solve or resolve the model.

Solution: Biodiversity can regrow/recover but takes very long and there is an asymmetry between the ease and speed at which biodiversity can be destroyed and at which it can be recreated. You could, for example, spell out an equation of motion where it is only possible to lose biodiversity

$$x_{t+1} = \max\{x_t \ , \ f(x_t) - g(c_t)\}$$

Note: Here f would measure the resilience, which is higher if there is more biodiversity left.

Or you could let biodiversity increase, but at a much slower rate than its potential destruction by altering the above to

$$x_{t+1} = \max\{x_t + \alpha(f(x_t) - g(c_t))\ ,\ f(x_t) - g(c_t)\}$$

for some $\alpha << 1$ (substantially smaller than unity).