ECON4925 Resource economics, Autumn 2009

Lecture 1: Optimization in a dynamic context

A typical dynamic optimization problem involving a resource used in quantity $x(t)$ at time t is

$$Max \int_0^N e^{-rt} \left[u(x(t)) - C(x(t))\right] dt$$

subject to various constraints (depending on the details of the problem).

In the lecture we shall discuss
- the interpretation of the discount rate r
- the panning horizon (finite versus infinite N)
- the interpretation of $[u(x(t)) - C(x(t))]$

Interpretation of $[u(x(t)) - C(x(t))]$

The resource is a consumer good

Two goods x and z, with $z = z_0 - C(x)$ and z_0 exogenous

$$U(x,z) \approx U(x,z_0) + U_z(x,z_0) \cdot (z - z_0)$$

Assuming that $U_z(x,z_0)$ is almost independent of x for the variations in x we are considering, we can normalize the utility function so that $U_z(x,z_0) = 1$ and rewrite the equation above as

$$U(x,z) = u(x) - C(x)$$

Notice that this implies $\frac{U_x}{U_z} = u'(x)$, so that $u'(x)$ can be interpreted as the price of the resource (in terms of the numeraire good z).

The resource is an input into production

There is an all-purpose good, produced with the resource as one of its inputs. The good may be used for consumption and as an input into producing the resource:

$$consumption = F(x,z) - C(x)$$

where z denotes other inputs, assumed exogenous. We can thus write consumption as $u(x) - C(x)$. Notice that this implies $F_x = u'(x)$, so that $u'(x)$ will be equal to the price of the resource (in terms of the consumption good) if the producer is a profit maximizing price taker.

Optimal control theory (rough sketch)

A typical dynamic optimization problem may be written as

$$\begin{aligned} V(S_0) &= \max \int_0^N e^{-rt} F(x(t), S(t)) dt \\ s.t.\ \dot{S}(t) &= G(x(t), S(t)) \\ x(t) &\geq 0 \\ S(0) &= S_0\ (exogenous) \\ S(N) &\geq 0 \end{aligned}$$

To solve we define the current value Hamiltonian:

$$H = F(x(t), S(t)) + \lambda(t) G(x(t), S(t))$$

Note: The interpretation of $\lambda(0)$ is

$$\lambda(0) = V'(S_0)$$

Necessary conditions for optimum (omitting time references):

$$\begin{aligned} \dot{\lambda} - r\lambda &= -\frac{\partial G}{\partial S} \\ \frac{\partial H}{\partial x} &= 0\ (for\ x > 0) \\ e^{-rN}\lambda(N) &\geq 0\ (=\ for\ S(N) > 0) \end{aligned}$$

(The last of these three conditions is called the transversality condition.)