JV; August 2016 – ECON 4925 (2016)

Lecture Note 1: Optimal Resource Extraction

("An attempt to get rid of some confusion from the lecture on August 31.")

Define the gross consumers' surplus from consuming x units of the resource as the area below the demand curve for x units, and given (within the partial framework of ours) by $U(x)$, when assuming $U' > 0, U'' < 0$ and $U'(0) = \infty$ (no choke price).

The planner's objective is to maximize present discounted value (PDV) of consumers' surplus, from today (t = 0) to infinity, with a utility discount rate i, subject to a resource constraint, known initial resource stock, S_0, and a "terminal" constraint as given by $\lim_{t\to\infty} S(t) \geq 0$. Resource extraction during a short interval of length dt, is $x(t)dt$, with $x(t)$ as the rate (per unit of time) of extraction at t . With $S(t) = S_0 - \int_0^t x(\tau)d\tau$ is the remaining reserve (a stock variable) at t, we can derive $\dot{S}(t) = \frac{dS(t)}{dt} = -x(t)$ as the rate of change in the remaining reserves at t, being equal to the rate of extraction.

The planner's problem is therefore:

$$Max_{x(\cdot)} \int_0^\infty e^{-it} U(x(t))dt \text{ subject to } \int_0^\infty x(t)dt \leq S_0 \text{ with } x \geq 0\ \forall t, \lim_{t\to\infty} S(t) \geq 0$$

We assume also that the planner is using the same discount factor as what the private agents are using. The issue we are dealing with is: How to divide a fixed cake (of known size) over an infinite number of generations?

Because we assume away extraction costs, the market price will coincide with the resource rent. Hence our optimization problem is:

$$Max_{x(\cdot)\geq 0} \int_0^\infty e^{-it} U(x(t))dt \ s.t.\ \dot{S}(t) = -x(t),\ S(t) \geq 0 \forall t \text{ and } S_0 > 0 \ .$$ [1]

[1] The integral is assumed to exist; if not, we have to reformulate our optimality criterion.

(Here we have replaced our resource constraint $\int_0^\infty x(t)dt \le S_0$ with the differential equation $\dot{S}(t) = -x(t)$.)

Suppose that we formulate the same problem in discrete time, and the objective is to:
$Max_{x_0, x_1, ...} \sum_{t=0}^{\infty} \beta^t U(x_t)$ $gitt$ $\sum_{t=0}^{\infty} x_t \le S_0$. Again, the goal is to maximize the PDV of future utility flows from consuming the resource, with the utility function having the same properties as above, and $\beta := \frac{1}{1+i}$ is the one-periodic discount **factor**, and i interpreted as a one-period utility discount **rate**. (Given our assumptions, this problem has a solution, with the resource constraint binding; hence there will exist

a positive Lagrange multiplier λ, and a corresponding Lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t U(x_t) - \lambda \left[\sum_{t=0}^{\infty} x_t - S_0 \right]$$

so that an optimal extraction path must obey: $\frac{\partial L}{\partial x_t} = \beta^t U'(x_t) - \lambda = 0$ for t =0,1,……,

because we will have positive extraction in all periods. The resource constraint will bind as $U'(0) = \infty$, and $\lambda > 0$, which is measured in utility units and is the present discounted shadow value of the resource (the opportunity cost). The optimality condition then tells us that the PDV of marginal utility (the PDV of the marginal benefit from consuming the resource as extracted) should be the same for all periods and equal to the opportunity cost, which is the PDV of the resource as unextracted.

If we go to continuous time, with continuous compounding, the optimality condition will be: $e^{-it}U'(x(t)) = \lambda(t) = \lambda$ (a positive constant).

As shown in Vislie (2016), cf. the last section on Dorfman's derivation, the optimality conditions can then be found from forming the PV Hamiltonian, as given by, $H(x, S, \lambda, t) = e^{-it}U(x) - \lambda x$, where $x \ge 0$ is a control variable, S is the state variable and λ a costate or adjoint variable.

An extraction path solving our problem must then obey:

(1) $\dfrac{\partial H}{\partial x} = e^{-it}U'(x(t)) - \lambda(t) = 0$

(2) $\dot{\lambda}(t) = -\dfrac{\partial H}{\partial S} = 0 \Rightarrow \lambda(t) = \lambda > 0$

where both derivatives are evaluated for the optimal solution, and the fact that in this case H does not involve the state variable. At last we have the transversality conditions, as given by

(3) $\lim_{t\to\infty} \lambda(t) \geq 0$ and $\lim_{t\to\infty} \lambda(t)S(t) = 0$

(These conditions will induce us not to leave any positive amount of the resource unused in the limit. Because $\lambda > 0$, the resource is fully depleted over the infinite planning horizon, as the last condition in (3) then is given by $\lambda \cdot \lim_{t\to\infty} S(t) = 0 \Rightarrow \lim_{t\to\infty} S(t) = 0$.)

Properties of the optimal solution are:

a) The resource is fully extracted; $\int_0^\infty x(t)dt = S_0$

b) The present value opportunity cost of extraction, λ, is constant (the no-arbitrage principle) so that the current opportunity cost or current rent is increasing at a rate equal to i; i.e. $\lambda(t) = \lambda(0)e^{it}$.

c) The extraction path is declining over time, as seen from differentiating (1) w.r.t. time: $-ie^{-it}U'(x(t)) + e^{-it}U''(x(t)) \cdot \dot{x}(t) = \dot{\lambda} = 0 \Leftrightarrow$ $-i\lambda + \lambda \dfrac{x(t)U''(x(t))}{U'(x(t))}\dfrac{\dot{x}(t)}{x(t)} = 0 \Leftrightarrow i + \hat{\omega}(x(t))\dfrac{\dot{x}(t)}{x(t)} = 0$, where we have defined the flexibility of the marginal utility (as the inverse of the intertemporal elasticity of consumption and being equal to the coefficient of intergenerational inequality aversion), as $\hat{\omega}(x) = -El_x U'(x) = -\dfrac{x}{U'(x)}U''(x) > 0$. We see that the extraction path is declining and will approach zero asymptotically, and with $(-\dfrac{\dot{x}(t)}{x(t)}) = \dfrac{i}{\hat{\omega}(x(t))}$, being steeper the higher is the discount rate (more impatience or less weight put on future generations – lower discount factor), with higher

$x(0)$, and a less steeper path the higher is $\hat{\omega}(x(t))$. A high value of this flexibility reflects a desire for "consumption smoothening", expressing the fact that "higher consumption from some level is **not** valued very much", as the marginal utility declines significantly if consumption should increase.

d) We can express demand from $U'(x(t)) = \lambda e^{it} \Rightarrow x(t) = D(\lambda e^{it})$, with $D' < 0$ due to strict concavity of U. Then the opportunity cost is determined from $\int_0^{\infty} D(\lambda e^{it})dt = S_0 \Rightarrow \lambda = g(i, S_0)$ with g differentiable. (In problem Set 2 you will be asked to show the sign of $\frac{\partial g}{\partial i}$ and $\frac{\partial g}{\partial S_0}$.)

e) If the current market price (here equal to rent) is derived from $U'(x) = p$, then the intertemporal competitive equilibrium will be optimal given the correct initial price, $p(0)$, along with the price (or rent) obeying the Hotelling Rule: $p(t) = p(0)e^{it}$. Hence, with a complete set of forward markets (perfect foresight) the intertemporal competitive equilibrium is optimal.

Alternatively we can formulate the problem by looking at the current value Hamiltonian, as given by $\hat{H}(x, S, \Lambda, t) = U(x) - \Lambda x$, where Λ is the *current* shadow value of the resource in units of utility.[2]

The optimal extraction at some point in time t, as seen from $t = 0$, should here be identical to what we would choose if we were to take a decision at *t* with the remaining resource at that time as a resource constraint. Hence we have $\Lambda(t) = \lambda e^{it}$. (Think if this equality is not satisfied.)

The solution in this problem must then obey:

$$(3) \quad \frac{\partial \hat{H}}{\partial x} = U'(x) - \Lambda = 0$$

$$(4) \quad \dot{\Lambda}(t) = -\frac{\partial \hat{H}}{\partial S} \Leftrightarrow \frac{d}{dt}\ \lambda e^{it} = \underbrace{\dot{\lambda}}_{=0} e^{it} + i\lambda e^{it} = 0 \Leftrightarrow \dot{\Lambda}(t) = i\lambda e^{it} = i\Lambda(t)$$

(Again the state variable does not enter $\hat{H}$.)

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[2] This was the cause for my confusion because I dropped the Current Value version of the Ramsey Model.

The transversality conditions will then be (see Michael's note) written as:

(5) $\lim_{t\to\infty} e^{-it}\Lambda(t) \geq 0$ and $\lim_{t\to\infty} e^{-it}\Lambda(t)S(t) = 0$

Because $\lambda(t) = e^{-it}\Lambda(t)$, we have , and the remaining resource stock must approach 0 asymptotically, and all the results above will carry over – of course.)