JV; October 2016 – ECON 4925

Lecture Note 3: Resource Extraction under Uncertainty[1]

This note is related to resource extraction under uncertainty. We will in addition to the examples presented in lectures, also give some illustrations that can be read out the papers about uncertainty on the reading list (Dasgupta et al. (1978), Hoel (1978) and Kumar (2005).) Not only is the topic itself important and interesting, but there are some interesting modeling techniques that might be helpful in other applications as well. The problems to be dealt with are:[2]

- Wealth management and stochastic foreign demand
- Extraction with unknown resource stock
- What is meant by saying that one distribution is more risky than another one?

a) Optimal savings in a resource-rich SOE facing uncertain future demand

The problem studied in this section is one that is analyzed (not completely) in Dasgupta et al. (1978)[3]. The full background of the story can be read out from their paper.

The Small Open Economy (SOE) under consideration is selling natural resource (oil) at the world market with some market power as given by the demand function $p(E)$. Instead of treating the problem as if demand is known with full certainty, we now assume uncertainty, due to the possible arrival of new technology that more or less will replace the natural resource for the buyers. We now assume that if and when this happens, will lead to a downwards shift in the foreign demand for oil. The size of the shift is known[4], but *we don't know when* the shift occurs. Hence the point in time when the country will face new demand conditions is stochastic, as seen from ex ante or at the outset of the planning period. Let this point in time be denoted T, at which demand shifts from one level to another, lower one, from $p(E)$ to $q(E) < p(E)$ for any $E \geq 0$,

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[1] Thanks to Anne Killi for comments and pointing to some errors and meaningless sentences.

[2] In addition you will get the answer to problem 2 in Seminar Set 3, where you were asked to analyze the issue of optimal saving in a small open resource-rich economy when the rate of return on foreign assets (or debt) is uncertain in the sense that you know it will drop, but you don't know when. The third bullet point has not been covered in lectures – it is provided here because the notion of "something being more risky than something else" should be part of your knowledge arsenal. The method presented here is very useful and necessary for reading some of the articles on the reading list.

[3] Partha Dasgupta, Robert Eastwood and Geoffrey Heal (1978), Resource Management in a Trading Economy, *Quarterly Journal of Economics*, 92 (2), pp. 297 – 306.

[4] What is relevant is the expected size of the shift.

indicating a lower marginal willingness to pay for oil after having introduced a substitute for oil.

Define then a prior probability distribution $G(\tau) = \Pr(T \le \tau)$, that is the probability for the shift to occur before some τ. This function is twice continuously differentiable, with positive density $dG(\tau) = g(\tau)d\tau$ for $\tau \in [0, \infty$. We divide the planning horizon in two disjoint intervals; $[0, \tau] \wedge \tau, \infty]$ for some realized value of T equal to τ. At this realized point in time we have some remaining reserves and national wealth as given by the state (S_τ, W_τ). Given this new initial values of the state variable for the continuation regime $[\tau, \infty$; from the point in time when the demand shifts downwards, we can define a value function, as the maximal PDV of future utility flows (the continuation payoff) as:

$$V(S_\tau, W_\tau) = \max_{(c,x,E,K)} \int_\tau^\infty e^{-r(t-\tau)} U(c(t)) dt$$

s.t.

$\dot{S}(t) = -(x(t) + E(t))$, S_τ given, and $\lim_{t\to\infty} S(t) \ge 0$

$\dot{W}(t) = F(K(t), x(t)) + \theta(W(t) - K(t)) + q(E)E - c(t)$; W_τ given, and $\lim_{t\to\infty} W(t) \ge 0$

This value function is, due to infinite horizon, independent of τ. (This solution is as if we just move the planning horizon from $t = 0$ to $t = \tau$.) We assume $U(0) = 0, U' > 0, U'' < 0, U'(0) = \infty, U'(\infty) = 0, F_K > 0, F_{KK} < 0, F_x > 0, F_{xx} < 0$ and $q'(E) \le 0$. We also assume that $\omega = -El_c U'(c)$ is constant.

The solution to this program is similar to what we have derived for the case with no uncertainty: the only difference is that the initial point in time is shifted outwards. From this date, we know what to do, by pursuing the policy as determined by the previously derived static and dynamic (no-arbitrage) conditions. The shadow values at the time when demand shifts, or for the new regime, is therefore given by $\lambda(\tau) := \dfrac{\partial V}{\partial S_\tau} = V_S$ and $\mu(\tau) := \dfrac{\partial V}{\partial W_\tau} = V_W$, both being positive.[5] These shadow values show the value in utils of having one more unit of the resource at the start of the new regime, or the increase in

[5] We should, more correct perhaps, have written $\lambda(\tau^+)$, so as to distinguish between «before» and “after”.

PDV of future utility from having a unit more wealth, or oil, at the start of the new regime.

For any realization of the date of demand shift we have total welfare as given by $\int_0^\tau e^{-rt}U(c(t))dt + e^{-r\tau}V(S_\tau, W_\tau)$. Because the date when demand shifts is stochastic we have to give a probability weight to each possible outcome. We can then put up the expected payoff as:
$\int_0^\infty g(\tau)\left[\int_0^\tau e^{-rt}U(c(t))dt + e^{-r\tau}V(S_\tau, W_\tau)\right]d\tau$, and the objective is to choose consumption, resource extraction and the allocation between domestic and foreign use, as well as investment in real capital, so as to maximize this expected payoff subject to:
$\dot{S}(t) = -(x(t) + E(t))$, $S(0) = S_0$ and also requiring $S(t) > 0 \ \forall t$ as long as no shift has occurred, and $\dot{W}(t) = F(K(t), x(t)) + \theta(W(t) - K(t)) + p(E)E - c(t)$, with $W(0) = \mathrm{W}_0$ and we could aim at a certain wealth target once the shift in demand occurs or just let wealth be whatever it is.[6] On integrating by parts, we can rewrite our objective function, when defining $R(\tau) := \int_0^\tau e^{-rt}U(c(t))dt$, and using that $G'(\tau) = g(\tau)$, $G(0) = 0, G(\infty) = 1$ and using Leibniz' rule, so that:

$$\int_0^\infty G'(\tau)[R(\tau) + e^{-r\tau}V]d\tau = G(\tau)R(\tau)\Big|_0^\infty - \int_0^\infty G(\tau)R'(\tau)d\tau + \int_0^\infty g(\tau)e^{-r\tau}V(\mathrm{S}_\tau, W_\tau)d\tau$$

$$= 1\cdot R(\infty) - 0\cdot R(0) - \int_0^\infty G(\tau)e^{-r\tau}U(c(\tau))d\tau + \int_0^\infty g(\tau)e^{-r\tau}V(\mathrm{S}_\tau, W_\tau)d\tau$$

$$= \int_0^\infty e^{-rt}[(1 - G(t))U(c(t)) + g(t)V(S_t, W_t)]dt$$

The goal is to maximize this function, subject to the state constraints above. Suppose that a solution exists and is unique. Then, as long as the shift has not occurred, the planner will pursue the solution from this problem, which is found from defining the CV Hamiltonian as:

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[6] Perhaps a more sensible strategy could be to require that we have to meet a wealth target; hence we might assume $W(t) \geq w$ as long as we have not yet experienced the shift in demand. This lower bound on wealth could itself be part of the optimization program. We don't pursue this idea further.

$$H = (1-G)U(c) + gV(S,W) + \eta\Big[F(K,x) + \theta(W-K) + p(E)E - c\Big] - \Lambda(x+E)$$

where (η, Λ) are current (measured in utils) shadow values of the two state variables, for the before-shift regime.

An (interior) optimal solution must then obey:

(1) $\dfrac{\partial H}{\partial c} = (1-G)U'(c) - \eta = 0 \Rightarrow U'(c(t)) = \dfrac{\eta(t)}{1-G(t)} := Q(t)$

where $Q(t)$ can be seen as a spot price on consumption, or a conditional price on consumption (in utils), conditional on not having had a shift by t. (When discounted back to $t = 0$, we then have the price paid today for one unit of consumption delivered at t, conditional on no shift has taken place by then.) Also, for an interior solution, we have:

(2) $\dfrac{\partial H}{\partial x} = \eta F_x - \Lambda = 0$

(3) $\dfrac{\partial H}{\partial E} = \eta\Big[p(E) + Ep'(E)\Big] - \Lambda = 0$

(4) $\dfrac{\partial H}{\partial K} = \eta\Big[F_K - \theta\Big] = 0$

(5) $\dot{\eta}(t) = r\eta(t) - \dfrac{\partial H}{\partial W} = r\eta(t) - g(t)\dfrac{\partial V}{\partial W} - \eta(t)\theta = [r-\theta]\eta(t) - g(t)\mu(t)$

(6) $\dot{\Lambda}(t) = r\Lambda(t) - \dfrac{\partial H}{\partial S} = r\Lambda(t) - g(t)\dfrac{\partial V}{\partial S} = r\Lambda(t) - g(t)\lambda(t)$

What conclusions can be drawn from these conditions?

First, from (2) and (3) we get that the condition for *static efficiency* is satisfied: For any *given* extraction rate, the optimal allocation between domestic use of oil and export, as well as domestic return on capital being equal to the rate of return on foreign assets:

(7) $F_x = \dfrac{\Lambda}{\eta} = p(E) + Ep'(E)$ and $F_K = \theta$

Secondly, *dynamic efficiency* is satisfied from the no-arbitrage condition:

(8) $\dfrac{d}{dt}\ln F_x = \dfrac{d}{dt}\ln\Big[p(E) + Ep'(E)\Big] = \dfrac{\dot{\Lambda}}{\Lambda} - \dfrac{\dot{\eta}}{\eta} = r - g(t)\dfrac{\lambda(t)}{\Lambda(t)} - \left[r - \theta - g(t)\dfrac{\mu(t)}{\eta(t)}\right]$

Hence we get:

$$(8)' \quad \frac{d}{dt}\ln F_x = \frac{d}{dt}\ln\left[p(E)+Ep'(E)\right] = \theta + g(t)\left[\frac{\mu(t)}{\eta(t)} - \frac{\lambda(t)}{\Lambda(t)}\right]$$

showing the rate of increase in the resource rent along an equilibrium path, as long as no shift has occurred. (Dasgupta et al. eliminate all interesting aspects of introducing uncertainty by assuming that $\frac{V_S}{V_W} := \frac{\lambda}{\mu} = \frac{\Lambda}{\eta}$, so that the relative shadow prices do not change when shifting regime, without any justification.)

If we consider F_x as the domestic price of the resource in units of output (which is also equal to the marginal revenue from exporting), the relative rate of change per unit of time in this price should be $\theta + g(t)\left[\frac{\mu(t)}{\eta(t)} - \frac{\lambda(t)}{\Lambda(t)}\right]$, where the last term can be regarded as risk-premium term for delaying extraction. The rate of increase in the resource rent, before the shift will in general be different from the after-shift rate of increase. The reason is that due to a negative shift in foreign demand, the marginal value of the resource will be *reduced* once we enter the new regime, along with a change in the shadow value of wealth.

To see the impact of this shift, let us consider the decision at any point in time as long as no shift has occurred before that point in time. Define therefore the conditional density at t, $\frac{g(t)}{1-G(t)} := h(t)$, or a hazard rate for the event to occur, in the very near future, given that it has not yet occurred.[7] Then we can rewrite (8)′ in a way that is easier to interpret:

$$(9) \quad \frac{d}{dt}\ln F_x = \theta + \frac{g(t)}{1-G(t)}\left[\frac{V_W(t)}{\frac{\eta(t)}{1-G(t)}} - \frac{V_S(t)}{\frac{\Lambda(t)}{1-G(t)}}\right] = \theta + h(t)\left[\frac{V_W(t)}{\frac{\eta(t)}{1-G(t)}} - \frac{V_S(t)}{\frac{\Lambda(t)}{1-G(t)}}\right] = \frac{d}{dt}\ln\left[p(E)+Ep'(E)\right]$$

We guess that the value of a marginal unit of oil after the shift, $V_S = \lambda$, is lower than the conditional shadow value as long as the shift has not occurred; hence the last term

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[7] The hazard rate is defined as $h(t) = \lim_{\Delta\to 0}\frac{\Pr(\tau \in \left[t, t+\Delta\right] \big| \tau > t)}{\Delta} = \frac{g(t)}{1-G(t)}$.

$\frac{V_S(t)}{\frac{\Lambda(t)}{1-G(t)}} = \frac{\lambda(t)}{\frac{\Lambda(t)}{1-G(t)}} < 1$. Also, the marginal value of a unit wealth after the shift is expected to be higher than the conditional marginal value of wealth as long as the shift has not yet occurred; hence we guess that $\frac{V_W(t)}{\frac{\eta(t)}{1-G(t)}} = \frac{\mu(t)}{Q(t)} > 1$.[8] It seems therefore in the planner's interest to transfer wealth from the before-shift regime to the after-shift regime, as the marginal value of wealth is lower in the before-shift regime. This transfer is of course implemented by increased saving or lower consumption in the before-shift regime. Hence, for any t for which the hazard rate is positive, the risk term in (9) is positive. Therefore, along a conditional strategy, the resource rent is increasing at a rate above θ. That also means that the price path is steeper than what it would have been with no uncertainty, with a lower initial price and higher initial extraction (both for domestic use, but mostly for export) so as to take advantage of facing a high demand abroad. ("Sell when the price is high; tomorrow might be too late".)

What will happen should the shift occur at point in time *t*?
Consider first consumption at that date; i.e., "just before and just after". Because we have that $U'(c(t^+)) = \mu(t^+) > Q(t^-) = U'(c(t^-))$, consumption rate jumps *downwards*, due to concavity of $U(c)$, with $c(t^+) < c(t^-)$. This seems reasonable because the shift makes us less wealthy as a nation; cf. the discussion above.

Let us, just for ease of exposition, assume that $\theta = r$. In that case we know that that the consumption path in the continuation (after-shift) regime, must obey $r + \omega \frac{\dot{c}(t)}{c(t)} = \theta$, from which it follows that $\frac{\dot{c}}{c} = 0 \ \forall t \in \ t^+, \infty$ with $\theta = r$. In this regime we have a constant consumption rate. On the other hand, in the before-shift regime we get, from (1) and (5), that $\ln U'(c(t)) = \ln \eta(\mathrm{t}) - \ln(1 - G(t))$. Differentiate this with respect to time yields:

[8] One way of rationalizing this statement is to think of a person, with a bank deposit, being unemployed and then facing a large income loss. The marginal value of the bank deposit is then higher as unemployed than as employed.

$$-\omega\frac{\dot{c}(t)}{c(t)}=\frac{\dot{\eta}(t)}{\eta(t)}-\frac{(-g(t))}{1-G(t)}=r-\theta-g(t)\frac{\mu(t)}{\eta(t)}+h(t)=r-\theta+h(t)-\frac{g(t)}{1-G(t)}\frac{\mu(t)}{\frac{\eta(t)}{1-G(t)}},$$

hence:

$$(10)\quad r-h(t)\frac{\mu(t)-Q(t)}{Q(t)}+\omega\frac{\dot{c}(t)}{c(t)}=\theta\Rightarrow\omega\frac{\dot{c}(t)}{c(t)}=h(t)\frac{\mu(t)-Q(t)}{Q(t)}\ \text{ if } r=\theta$$

Hence, given our assumptions, the conditional consumption path, conditional on not having experienced a shift, is increasing.
Due to future risk of facing a lower foreign demand for oil, the pure (risk-free) rate of impatience r is adjusted downwards, because of $Q<\mu$. The term $\frac{\mu-Q}{Q}$ can be seen as an interest rate, showing the rate of return in the following way: The marginal rate of substitution between consumption before and after the shift is $\frac{\mu}{Q}$, showing the number of before-shift consumption units one is willing to give up ("save") to have one additional consumption unit after the shift, conditional on not yet having had a shift, which can happen in the very near future with density $h(t)$. Therefore, we can interpret $\frac{\mu}{Q}-1=\frac{\mu-Q}{Q}$ is a "risk-adjusted required rate of return on deferring consumption".
Because we want to prepare for a less wealthy future, the overall required rate of return from saving as long as no shift has yet occurred, is reduced, as compared to the risk-free case. The risk-adjustment can then be seen as lowering the rate of impatience; more weight is therefore put on the future (as long as we have not yet had a shift).

How is extraction affected by the possibility of facing a lower demand from foreign buyers? We know that *after* the shift we must have:

$$(11-i)\quad F_x=\frac{\lambda}{\mu}=q(E)+Eq'(E)\quad \text{Static efficiency}$$

$$(11-ii)\quad \frac{d}{dt}\ln F_x=\frac{d}{dt}\ln\left[q(E)+Eq'(E)\right]=\theta=F_K\qquad \text{Dynamic equilibrium}$$

The lower foreign demand will lead to less oil sold abroad than what was the case before the shift.[9] What then about domestic use? As long as no shift has happened (i.e., before; hence superscript B), we have from (7) that $F_x^B = \dfrac{\Lambda}{\eta} = p(E) + Ep'(E)$.

Concentrating on the first equality we then observe that:

$$(12) \quad F_x^B = \frac{\Lambda}{\eta} = \frac{\frac{\Lambda}{1-G}}{\frac{\eta}{1-G}} = \frac{\frac{\Lambda}{1-G}}{Q} > \frac{\lambda}{Q} = \frac{V_S}{Q} > \frac{V_S}{\mu} = \frac{V_S}{V_W} = \frac{\lambda}{\mu} = F_x^A$$

The first inequality follows from $V_S = \lambda < \dfrac{\Lambda}{1-G}$, as a marginal unit of oil has a higher value before the shift than after, whereas the second inequality follows from $V_W = \mu > \dfrac{\eta}{1-G} := Q$. With decreasing marginal productivity of oil in domestic production, we (normally, but there might be some modifying effect through the variation in the other input) will have that $x^A > x^B$. After the shift we use more oil at home to produce more consumption goods to replace imports, when facing a loss of export revenue.

b) Extraction with unknown resource stock

We will now consider the problem which can be regarded as one of eating a cake of unknown size. We considered as a starter a two-period cake-eating problem with uncertain resource stock that we learnt the true value of at the beginning of the second (last) period. In general we then have the ex ante optimality condition:

$$(13) \quad U'(x_1) = \beta \int_0^\infty U'(s - x_1) f(s) ds := \beta E_S U'(S - x_1)$$

where x_1 is extraction (equal to consumption) in period 1; S the stochastic initial deposit with density function $dF(s) = f(s)ds$ for $s \in 0, \infty$, β a one-period discount factor and E the expectations operator. For an interior solution, optimality requires that current marginal utility is set equal to PDV of expected marginal utility of consumption in the last or second period.

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[9] The following line of reasoning might need to be modified by taking the cross effects into account. As the choice of *K* is affected by the shift, the marginal productivity of oil in domestic production might be affected in a way not being accounted for in the text.

We will consider the continuous time version of this problem by having an initial reserve that is considered as stochastic. However, as time elapses, we learn more about the size as we are reaching higher and higher levels of accumulated extraction. (Learning means that we truncate the distribution, because as long as we have not yet hit the bottom, the initial deposits are exceeding the cumulated extraction till then.)

We have a standard felicity or utility function $U(x(t))$, with consumption rate at t being equal to the rate of extraction (consumption) as given by $x(t)dt$ during a short interval of time, of length dt. We ignore extraction costs. The welfare criterion is again expected PDV of future felicities; i.e., the expected value of $\int_0^\infty e^{-rt}U(x(t))dt$ subject to a standard "take-out-rule"; $\dot{S}(t) = -x(t)$, with $S(t) = S_0 - \int_0^t x(\tau)d\tau$, as remaining reserves, but where the initial stock now is not known, but considered as stochastic. We will consider how the competitive solution is affected by this type of uncertainty. In addition we keep the standard assumptions: $U(0) = 0, U' > 0, U'' < 0, U'(0) = \infty \,\&\, U'(\infty) = 0$, with r a positive felicity rate or utility discount rate. The choice of an extraction plan is chosen without having the exact information about when the plan will lead to full exhaustion.

Under full certainty we have that the optimal solution is characterized by $U'(x(t)) = \lambda$ and $\int_0^\infty x(t)dt = S_0$ with $\lambda(t) = \lambda(0)e^{rt}$ as the current value shadow price of the remaining reserve.[10] This solution can be realized as a (partial) competitive equilibrium, with demand as given by $U'(x(t)) = p(t)$, with the social discount rate being equal to the one used by private agents in the competitive economy. To have an intertemporal equilibrium with demand being satisfied at any instant of time, the price (here equal to the resource rent) must increase at a rate equal to the rate of discount, with $p(t) = p(0)e^{rt}$. If the buyers are on their demand curve, then we have market clearing as $U'(x(t)) = p(0)e^{rt}$. Then the competitive solution will coincide with the optimal solution, as long as we have a full set of forward markets, with the initial price set so as to meet the resource constraint. Within this context price is monotonically increasing over time, whereas extraction rate is monotonically decreasing, approaching zero asymptotically.

[10] See Lecture Note 1: Optimal Resource Extraction, by JV; August 2016.

Let us now see how the solution under full certainty will be modified if the initial reserve is not known with full certainty. Suppose we have some geological information saying that the prior or ex ante probability distribution for the initial reserve, S_0, is given by the cumulative distribution $G(s) = \Pr(S_0 \leq s)$, defined for $s \in [0, \infty$, continuously differentiable, with a density function a given by $dG(s) = g(s)ds$. (We could have introduced a positive lower and upper bound on the set of possible reserves, say $[\underline{s}, \overline{s}]$, because we know for sure that it will not be lower than $\underline{s}$ and not above $\overline{s}$. We also assume that the expected as well as the variance of this stochastic variable exist and are finite.) As we have reached some level of cumulated extraction (which of course will depend on the previous extraction program), we can calculate the conditional distribution for the remaining reserve, and use this information to find the proper balancing between current and future extraction. (As always, at any point in time, we have to balance the expected cost of delaying extraction with the expected benefit from delaying.)

Therefore, we can now transform this problem into the following "unknown time"-problem, in a similar way as been done in Kumar's paper.[11] Define therefore the accumulated extraction at some point in time as $Y(t) = \int_0^t x(\tau)d\tau = S_0 - S(t)$, with $\dot{Y}(t) = x(t) = -\dot{S}(t)$. Define the point in time when accumulated extraction has reached a level *s*; according to $Y(t) = s$, as $\varphi(s)$. The function $Y(t) = s$ is monotonically increasing; hence it has an inverse and given by $\varphi(s) = Y^{-1}(s)$, which is increasing in *s*. (It's like filling up a tank from below with water and define the point in time the level has reached the height or level s.)

Define then the stochastic function $T = \varphi(S_0)$, where *T* now is the (stochastic) point in time of complete exhaustion or terminal date when we have full depletion of the stock, which in itself is affected by our previous extraction program. We can derive the induced distribution for the new stochastic variable *T* in the following way, by using integration by substitution or change of variable:

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[11] See Ramesh C. Kumar (2005), How to eat a cake of unknown size: A reconsideration, *Journal of Environmental Economics and Management*, 50 (2), pp. 408-421.

$$\Pr(S_0 \le Y(t)) = G(Y(t)) = \int_0^{Y(t)} g(s)ds = \int_0^t g(Y(v))\dot{Y}(v)dv = \Pr(T \le t) := \int_0^t \Omega'(v)dv = \Omega(t)$$

where we have defined a probability distribution for the point in time when we have reached complete depletion.

For any realized point in time τ when the reserves are exhausted, with continuation payoff equal to zero (once there are no more reserves to consume), the payoff is $\int_0^{\tau} e^{-rt}U(x(t))dt$ with $S_0 = s_0 = \int_0^{\tau} x(t)dt$. Because τ is stochastic, we have to weight each outcome with the corresponding density to get to the expected payoff, given by:

$$(14) \quad E_{\tau}\left[\int_0^{\tau} e^{-rt}U(x(t))dt\right] = \int_0^{\infty} \Omega'(\tau)\left[\int_0^{\tau} e^{-rt}U(x(t))dt\right]d\tau$$

The objective is to maximize this integral subject to $\dot{Y}(t) = x(t)$ and $Y(0) = 0, Y(t) \ge 0 \ \forall t$.

On integrating by parts the objective function can be expressed as:[12]

$$\int_0^{\infty} \Omega'(\tau)\underbrace{\left[\int_0^{\tau} e^{-rt}U(x(t))dt\right]}_{:=\Psi(\tau)}d\tau = \Omega(\tau)\Psi(\tau)\Big|_0^{\infty} - \int_0^{\infty} \Omega(\tau)\Psi'(\tau)d\tau = \Omega(\infty)\Psi(\infty) - \Omega(0)\Psi(0)$$

$$-\int_0^{\infty} \Omega(\tau)e^{-r\tau}U(x(\tau))d\tau = \int_0^{\infty} e^{-rt}U(x(t))dt - \int_0^{\infty} \Omega(t)e^{-rt}U(x(t))dt = \int_0^{\infty} (1-\Omega(t))e^{-rt}U(x(t))dt$$

$= \int_0^{\infty} (1-G(Y(t)))e^{-rt}U(x(t))dt$, because we have, by definition, $\Omega(t) = G(Y(t))$.

Again the problem is to determine an optimal strategy or a contingency plan for the period as long as exhaustion has not yet occurred or as long as we have not hit the bottom of the oil field.

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[12] This is exactlyt the same procedure as we used on page 3; but here we have $V = 0$, as the only difference.

The stochastic optimization problem has been turned into a deterministic control problem, of maximizing the new functional above, subject to $\dot{Y}(t) = x(t)$, with the control variable, x, being piecewise continuous, and a state variable $Y(t) \geq 0 \; \forall t$, with initial constraint $Y(0) = 0$.

The Solution

The CV Hamiltonian of this problem, with λ being a negative costate variable, is:

$$H(Y, x, \lambda, t) = (1 - G(Y))U(x) + \lambda x$$

An optimal solution, as long as we extract, must obey:[13]

$$(15-i) \quad \frac{\partial H}{\partial x} = (1 - G(Y))U'(x) + \lambda = 0$$

$$(15-ii) \quad \dot{\lambda} = r\lambda - \frac{\partial H}{\partial Y} = r\lambda + U(x)G'(Y) = r\lambda + g(Y)U(x)$$

One transversality condition for our problems is $\lim_{t\to\infty} e^{-rt}\lambda(t) = 0$.[14] Once we exhaust the stock, extraction and consumption drops discontinuously to zero, from a level where $\lambda < 0$.

Define a new shadow price $\Lambda = -\lambda$ as the one for the remaining stock, with a relationship corresponding to $(15-ii)$, as given by

$$(15-iii) \quad -\dot{\Lambda} = -r\Lambda + U(x)g(Y) \Leftrightarrow \dot{\Lambda} = r\Lambda - U(x)g(Y)$$

As long as the exhaustion-event has not happened, we adopt the following decision rule:

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[13] A minor, but important point is: These conditions will be sufficient if $\hat{H}(Y, \lambda, t) = \max_{x \geq 0} H(Y, x, \lambda, t)$ is concave in Y, for given values of λ and t. This is called Arrow's sufficiency theorem. We have $\hat{H}(Y, \lambda, t) = (1 - G(Y))U(x(Y, \lambda, t)) + \lambda x(Y, \lambda, t))$, with $x(Y, \lambda, t)$ as the maximizing choice, with $x_Y = \dfrac{G'(Y)U'}{(1 - G(Y))U''} < 0$, and with $\hat{H}_{YY}(Y, \lambda, t) = -G''(Y)U' - G'U''x_Y \leq 0$ if $G'' \geq 0$. Hence, in order to guarantee that we have an optimal solution, we should in principle restrict attention to distribution functions having density that is not "declining too fast".

[14] This condition, which is not necessary, is stronger than imposing $\lim_{t\to\infty} e^{-rt}\lambda(t) \geq 0$ and $\lim_{t\to\infty} e^{-rt}\lambda(t)S(t) = 0$.

$$(16) \quad U'(x(t)) = \frac{\Lambda}{1-G(Y)} \quad \& \quad r\Lambda = \dot{\Lambda} + g(Y)U(x)$$

What kind of trade-offs can be read out of these conditions? The first condition in (16) says that at some point in time we still extract, the market clearing spot price, $U'(x(t))$ is equal to $\frac{\Lambda}{1-G(Y)}$, where the unconditional shadow price, Λ, must obey the second condition in (16), as long as extraction takes place. However what matters is the dynamics of the conditional shadow price $\frac{\Lambda}{1-G(Y)}$.

First we want to see how the conditional extraction path evolves over time. Differentiating the first part of (16), written as $(1-G(Y(t)))U'(x(t)) - \Lambda(t) = 0$ while using the second condition as well, yields:

$$(17) \quad -g(Y(t))\dot{Y}(t)U'(x(t)) + (1-G(Y(t)))U''(x(t))\dot{x}(t) - \dot{\Lambda}(t) = 0$$

Use $\dot{Y}(t) = x(t)$ and define the hazard rate function $h(Y) = \frac{g(Y)}{1-G(Y)}$ as

$h(y) = \lim_{\Delta y \to 0} \frac{\Pr(Y \in \left[y, y+\Delta y \middle| Y \geq y\right]}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{\Pr(Y \in \left[y, y+\Delta y\right]}{\Pr(Y \geq y)}}{\Delta y} = \frac{g(y)}{1-G(y)}$, as the (conditional) probability that the remaining reserve at t is in $\left[y, y+\Delta\right]$ given that we have extracted an amount y at t. (Or one might define the hazard rate for hitting the terminal point in some interval $t, t+dt$, given that we have survived by t, as $\gamma(t)dt := \frac{G'(y(t))x(t)dt}{1-G(t)} = \frac{\Omega'(t)dt}{1-\Omega(t)} = h(y)x(t)dt$.)

On dividing through (17) by $1-G(Y)$, while using the second part of (16), we get:

$$-\frac{g(Y)}{1-G(Y)}xU'(x) + U''(x)\dot{x} - \frac{r\Lambda - g(Y)U(x)}{1-G(Y)} = 0 \Rightarrow$$

$$(18) \quad U''(x)\dot{x} = h(Y)xU'(x) + r\underbrace{\frac{\Lambda}{1-G(Y)}}_{-U'(x)} - h(Y)U(x) = rU'(x) + h(Y)\left[xU'(x) - U(x)\right]$$

On dividing through by $U'(x)$ to get: $\underbrace{\frac{U''x}{U'}}_{=-\omega}\frac{\dot{x}}{x}=r+h(Y)\frac{xU'(x)-U(x)}{U'(x)}$ so that the conditional extraction path must obey:

$$\text{(19)}\qquad \frac{\dot{x}(t)}{x(t)}=\frac{\left[\frac{\frac{U(x(t))}{x(t)}-U'(x(t))}{U'(x(t))}\right]h(Y(t))x(t)-r}{\omega(x(t))}$$

We note that it is not at all obvious that the extraction path is everywhere declining as was the case with full certainty and concave utility function. The first term is always positive with a strictly concave utility function. Hence if this term exceeds the rate of discount, the extraction path is (locally) an increasing function of time. (Here we can interpret $\omega(x)=-\frac{U''x}{U'}$ as the coefficient of relative risk aversion.) Also the hazard rate for hitting the terminal point in time just after we have survived t, $h(Y(t))x(t)dt$, when following the optimal consumption program, is playing a role. We come back to this growth rate later.

If we define the competitive spot price (which is here equal to the conditional opportunity cost or resource rent) as $p(t)=U'(x(t))$, we find the corresponding dynamics of this price from $\dot{p}=U''(x)\dot{x}\Rightarrow\frac{\dot{p}}{p}=\frac{U''x}{U'}\frac{\dot{x}}{x}=\underbrace{-\omega(x)}_{neg}\frac{\dot{x}}{x}$. Hence, from (19 we have:

$$\text{(20)}\qquad \frac{\dot{p}(t)}{p(t)}=r-\left[\frac{\frac{U(x)}{x}-U'(x)}{U'(x)}\right]h(Y)x\leq r$$

Along an optimal extraction path, conditional on not having reached a state of exhaustion, the resource rent or spot resource price (cf. the Hotelling Rule) is increasing at a rate **below** the rate of discount. Hence the initial price is set **above** the price that would have been set under no uncertainty (with reserves being equal to expected reserves in the present case). This makes sense. As long as we don't know the exact remaining resource reserve we want to impose a precautionary extraction program, by choosing lower extraction (at least) in the initial part of the planning period, until we get more information by following the conditional optimal program. As we reach higher and higher levels of accumulated amounts of resources without having reached the bottom, we learn more and receive “good news” that will affect current extraction. If it should be the case that we learn that reserves are so big that we “never” reach exhaustion, then as we get this information, the price must change in a manner that makes extraction higher than under certainty in the later phases of the extraction program.

Rewriting (20) yields the no-arbitrage condition:

$$(21) \qquad rp = \left[\frac{U(x)}{x} - U'(x)\right] \cdot h(Y)x + \dot{p} \Leftrightarrow r = \frac{\dfrac{U(x)}{x} - U'(x)}{U'(x)} h(Y)x + \frac{\dot{p}}{p}$$

Consider the first one at some point in time as long as exhaustion has not yet occurred. The LHS of the first expression is the cost of saving a marginal unit or deferring extraction, whereas the RHS is the (conditional) expected benefit from waiting or from having a marginal higher reserve, as given by the standard capital gain $\dot{p}$, as well as a term that reflects the expected benefit from a *longer duration of the extraction program*.

Let us explain this second term in some detail.[15] Suppose we reduce extraction by one unit at some point in time t when we still have some reserves left. The immediate or current loss in utility is $U'(x(t))$. What is the marginal benefit from lowering extraction at *t*? The reduced extraction will lead to pushing the date of exhaustion further into the future. Suppose that the date of exhaustion would have happened at $\tau > t$. With a constant extraction rate, x_τ, between these points in time, this extra unit we have saved in ground, can be spread over $\left[t, \tau\right]$, according to

$1 = \int_t^\tau x_\tau d\tau = x_\tau \int_t^\tau d\tau = x_\tau \left[\tau - t\right] \Rightarrow \tau - t = \frac{1}{x_\tau}$, showing the increased duration of the extraction program induced by a unit reduction in extraction at *t*. Because the utility rate (per unit of time) from consuming x_τ units is $U(x_\tau)$, the utility gain over the period $\left[t, \tau\right]$, is the utility rate multiplied by the length of the period; i.e., $U(x_\tau) \cdot \left[\tau - t\right] = \frac{U(x_\tau)}{x_\tau}$.

[15] This interpretation was provided by G.C. Loury, in “The Optimal Exploitation of an Unknown Reserve”, *Review of Economic Studies,* October 1978, pp.621-636.

In continuous time, the net benefit from delaying extraction at *t* will then be $\frac{U(x(t))}{x(t)} - U'(x(t))$. Dividing this net benefit by the opportunity cost, $U'(x(t))$, we get a rate of return. As seen from *t*, the conditional density for exhaustion in $\left[t, \tau\right]$ is $\frac{g(Y(\tau))x_{\tau}(\tau - t))}{1 - G(Y(t))}$. Hence the expected rate of return from delaying extraction at *t* by one unit, is, in continuous time, given by $\frac{\frac{U(x(t))}{x(t)} - U'(x(t))}{U'(x(t))} h(Y(t))x(t)$, which is one of the terms on the RHS of (21), and exactly equal to the first term in the numerator of (19).

But then we should be able to make a nice connection between the expected rate of return from delaying extraction (a saving decision in an economy without production) and the Keynes-Ramsey condition for optimal saving in an economy with production.

If consumption c is replacing "our" extraction rate x, we know that optimal saving (or optimal consumption) is characterized by $\frac{\dot{c}}{c} = \frac{F'(K) - \delta - r}{\omega(c)}$, where $F'(K) - \delta$ is the net rate of return in our one-commodity economy; cf. Vislie (2016). In our formula above, for the economy **without** production, the term $\left[\frac{\frac{U(x)}{x} - U'(x)}{U'(x)}\right] h(Y)x$ is "replacing" the net marginal rate of return from investment (delayed consumption or saving); as given by $F'(K) - \delta$. Above we have identified the term within brackets as the **rate of return** from delayed extraction – making the expected duration of the program longer, and this return is captured by a conditional density per unit of time as given by $h(Y)x$.

Hence written in the Keynes-Ramsey language, we have: The required rate of return from not extracting (not consuming) is equal to $r + \omega\frac{\dot{c}}{c}$. This required rate of return must match or balance the expected rate of return per unit investment from not extracting (not consuming, or delaying consumption), as given by $\left[\frac{\frac{U(x)}{x} - U'(x)}{U'(x)}\right] h(Y)x$.

Hence, the gain from delaying consumption or extraction is simply the gain from extending the expected duration of the extraction program. If the expected gain from waiting at some point in time is higher than the pure rate of impatience or utility rate of discount, extraction will increase or the resource price will decline. This is seen directly from (21) which, according to Loury, can be interpreted as a standard asset equilibrium condition: For each asset, the sum of "rental rate", $\dfrac{\frac{U(x)}{x} - U'(x)}{U'(x)} h(Y)x$, and the instantaneous rate of capital gain, $\dfrac{\dot{p}}{p}$, must be equal to the rate of interest (here rate of discount). If the conditional probability for exhaustion is very high, so that the rental rate exceeds the rate of discount, equilibrium will require a *negative* instantaneous capital gain, making the resource price declining and inducing an increase in extraction. As we follow some precautionary extraction strategy, obeying (21), taking into account the possibility of complete exhaustion, then as we learn that the reserves are larger than what we believed earlier, the resource price will decline. The reason is similar to the impact on the resource rent of a higher initial reserve; see Lecture Note 1 and Problem Set 2. (The higher is the stock the lower is the (initial) resource rent.)

Finally we can use (16) to express the decision rule at any point in time (as long as we still have some reserves left). We can now find a solution for $\Lambda(t)$ from the differential equation in (16), when making use of the strong transversality condition $\lim_{t\to\infty} e^{-rt}\Lambda(t) = 0$, when applying standard formula for solving a differential equation of the form $\dot{y}(t) + ay(t) = b(t)$. We then get, along with the "terminal" constraint, $\Lambda(t) = \int_t^\infty e^{-r(\tau - t)} U(c(\tau)) g(Y(\tau)) d\tau$. Inserting this into the first condition in (16), we get our decision rule:

$$(23) \quad U'(x(t)) = \int_t^\infty e^{-r(\tau-t)} \frac{U(x(\tau))}{x(\tau)} \frac{g(Y(\tau))x(\tau)}{1 - G(Y(t))} d\tau := E_\tau\left[e^{-r(\tau-t)} \frac{U(x(\tau))}{x(\tau)} \Big| \tau \geq t \right]$$

As long as exhaustion has not yet occurred, the current marginal utility of consumption must be balanced against the expected marginal cost, as given by the PDV (discounted to *t*) of average utility at the end of the program, with expectation taken over the induced conditional distribution for the terminal date τ, given that $\tau \geq t$.[16]

[16] See Proposition 1 in Loury (op.cit.)

c) What is meant by "one distribution being more risky than another one?"[17]

In a two-period model of resource extraction with no extraction costs but with resource stock being stochastic, and with the true value being learnt after the first period, we have that first period optimal extraction x_1 is determined from the following condition:

$U'(x_1) = \beta \int_0^{\infty} U'(s - x_1) f(s) ds = \beta E_S U'(S - x_1)$, where initial reserve S is distributed according to a cumulative distribution $F(s) = \Pr(S \le s)$ with density $dF(s) = f(s)ds$. (The function U, as usual, assumed to be strictly increasing and strictly concave.)

We want to consider a so-called *mean-preserving-increase in risk* (*MPS*) by looking at the following question: Let the distribution function also depend on a parameter γ, so that $F(s;\gamma) = \int_0^s f(v;\gamma) dv$, where γ is defined, say, on $[0,1]$. We also assume that more mass in the upper part of the distribution is considered better ("more is better than less"). We need some definitions.

Definition 1: A distribution $F(s,\gamma_1)$ dominates the distribution $F(s;\gamma_2)$ in a *first-order* stochastic sense, *FSD*, if and only if $F(s;\gamma_1) \le F(s;\gamma_2)$; with $F(0;\gamma) = 0$ and $\lim_{s \to \infty} F(s;\gamma) = 1$. (The distribution function $F(s;\gamma_1)$ is located below the other for all positive, finite values of s.)

Then we have some useful results:

Result #1: An agent with an increasing and differentiable von Neumann Morgenstern utility function, u, will prefer the distribution $F(s;\gamma_1)$ to $F(s;\gamma_2)$, according to expected utility, as we then have $\int_0^{\infty} u(s) dF(s;\gamma_1) \ge \int_0^{\infty} u(s) dF(s;\gamma_2)$.

[17] The results in Michael Hoel's paper *"Resource extraction, uncertainty, and learning"*; Bell Journal, 1978, can be used to discuss how we can analyze the impact on first-period extraction of a more risky distribution for S, by using a version of the method presented in this section.

Proof: Suppose the opposite; $\int_0^\infty u(s)dF(s;\gamma_2) - \int_0^\infty u(s)dF(s;\gamma_1) \geq 0$, saying that expected utility for the γ_2-distribution is higher than the one for the γ_1- distribution. Integrating by parts when using the boundary conditions $F(0;\gamma) = 0$ and $\lim_{s\to\infty} F(s;\gamma) = 1$, we get:

$$\int_0^\infty u(s)f(s;\gamma_2)ds - \int_0^\infty u(s)f(s;\gamma_1)ds$$
$$= u(s)F(s;\gamma_2)\Big|_0^\infty - \int_0^\infty u'(s)F(s;\gamma_2)ds - u(s)F(s;\gamma_1)\Big|_0^\infty + \int_0^\infty u'(s)F(s;\gamma_1)ds$$
$$= \int_0^\infty \left[F(s;\gamma_1) - F(s;\gamma_2)\right]u'(s)ds$$

Given our assumption, this expression should be positive. However, because of *FSD*, this must be negative; hence a contradiction and Result #1 is proved.

Another definition is the following: A distribution $F(s;\gamma_1)$ dominates $F(s;\gamma_2)$ in a *second-order* stochastic sense, *SSD*, if and only if $\int_0^y \left[F(s;\gamma_1) - F(s;\gamma_2)ds\right] \leq 0$ for any $y \geq 0$, with strict inequality for some values of *s* with positive densities. We then have:

Result #2: An agent with a twice differentiable, increasing and concave von Neumann Morgenstern utility function, $u(s)$, will prefer the distribution $F(s;\gamma_1)$ to $F(s;\gamma_2)$ if $F(s;\gamma_1)$ stochastically dominates in a second order sense $F(s;\gamma_2)$, according to expected utility $\int_0^\infty u(s)dF(s;\gamma_1) \geq \int_0^\infty u(s)dF(s;\gamma_2)$.

Proof: On using Result #1, and integrate by parts, as well as using Leibniz' Rule, we have:

$$\int_0^\infty u(s)dF(s;\gamma_1) - \int_0^\infty u(s)dF(s;\gamma_2) = -\int_0^\infty \left[F(s;\gamma_1) - F(s;\gamma_2)\right]u'(s)ds$$

$$= -\int_0^\infty u'(s)\left\{\frac{d}{ds}\int_0^s \left[F(v;\gamma_1) - F(v;\gamma_2)\right]dv\right\}ds = -u'(s)\int_0^s [F(v;\gamma_1) - F(v;\gamma_2)]dv\Big|_0^\infty$$

$$+\int_0^\infty u''(s)[\int_0^s [F(v;\gamma_1) - F(v;\gamma_2)]dv]ds = -\lim_{s\to\infty} u'(s)\int_0^s [F(v;\gamma_1) - F(v;\gamma_2)]dv$$

$+\int_0^\infty u''(s)[\int_0^s [F(v;\gamma_1) - F(v;\gamma_2)]dv]ds = \int_0^\infty u''(s)[\int_0^s [F(v;\gamma_1) - F(v;\gamma_2)]dv]ds \geq 0$, when using concavity of *u* and *SSD*.

Let us now use this second result to consider or rank two distributions, with the *same mean* or same *expected value*, but where one distribution is considered more risky than the other in *SSD*-sense, a so-called "*mean-preserving increase in risk*". The original distribution is $F(s;\gamma_1)$, and from this we construct a new one, called $F(s;\gamma_2)$, by taking some mass from "the middle of the original distribution" and put the mass in the tails, so that the two distributions have the same mean. If we let $\gamma_2 > \gamma_1$, and consider a small increase in γ, then this is similar to saying that the following two conditions must hold:

- $\int_0^\infty \frac{\partial F(s;\gamma)}{\partial \gamma} ds = 0$ (The mean is unchanged as we change γ.)
- $\Psi(y;\gamma) := \int_0^y \frac{\partial F(s;\gamma)}{\partial \gamma} ds \geq 0$ for some $y \in 0,\infty$ (This is *SSD*)

Let us then go back to our original two-period problem: We have one distribution for the size of initial reserves; $F(s;\gamma_1)$, and we have a first-period extraction, being dependent on the parameter γ_1, as determined from:

$U'(x_1(\gamma_1)) - \beta\int_0^\infty U'(s - x_1(\gamma_1))f(s;\gamma_1)ds = 0$. As we increase γ we get a more risky distribution. Define expected payoff as $V(\gamma) = U(x_1) + \beta\int_0^\infty U(s - x_1)f(s;\gamma)ds$, which is concave in x_1 as seen from $U''(x_1) + \beta\int_0^\infty U''(s - x_1)f(s;\gamma)ds < 0$, due to strict concavity of U (or risk aversion). What do we then know?

First we know that $V(\gamma_1) \geq V(\gamma_2)$, as any risk-averse agent $(U'' < 0)$ will prefer the less risky distribution; cf. Result #2.

Secondly, from the first-order condition, $U'(x_1(\gamma)) - \beta\int_0^\infty U'(s - x_1(\gamma))f(s;\gamma)ds = 0$, we can find the sign of $\frac{dx_1(\gamma)}{d\gamma}$. Differentiation yields:

$$U''(x_1(\gamma))\frac{dx_1(\gamma)}{d\gamma} - \beta\int_0^\infty U''(s - x_1(\gamma))(-\frac{dx_1(\gamma)}{d\gamma})\frac{\partial F(s;\gamma)}{\partial s}ds - \beta\int_0^\infty U'(s - x_1(\gamma))\frac{\partial^2 F(s;\gamma)}{\partial s\partial\gamma}ds = 0$$

Hence we find:

$$\text{(24)} \quad \frac{dx_1(\gamma)}{d\gamma} = \frac{\beta\int_0^\infty U'(s - x_1)\frac{\partial^2 F(s;\gamma)}{\partial s\partial\gamma}ds}{U''(x_1) + \beta\int_0^\infty U''(s - x_1)f(s,\gamma)ds}$$

The sign of the term in the denominator is negative, due to strict concavity of the V-function. Then we have: $sign\frac{dx_1(\gamma)}{d\gamma} = -sign\ \beta\int_0^\infty U'(s - x_1)\frac{\partial^2 F(s;\gamma)}{\partial s\partial\gamma}ds.$

Let us see what will determine the sign of this integral so that we can see under what circumstances more risk will lead to lower (or higher) first-period extraction. On using a theorem from Diamond and Stiglitz[18], the sign of this will follow from the sign of the third derivative, U'''.

Applying integration by parts, while using that $F(0;\gamma) = 0$ and $F(\infty;\gamma) = 1$ as well as $\frac{\partial F(0;\gamma)}{\partial\gamma} = \frac{\partial F(\infty;\gamma)}{\partial\gamma} = 0$, on the integral $\int_0^\infty U'(s - x_1)\frac{\partial^2 F(s;\gamma)}{\partial s\partial\gamma}ds$, yields:

$$\int_0^\infty U'(s - x_1)\frac{\partial^2 F(s;\gamma)}{\partial s\partial\gamma}ds = U'(s - x_1)\frac{\partial F(s;\gamma)}{\partial\gamma}\bigg|_0^\infty - \int_0^\infty U''(s - x_1)\frac{\partial F(s;\gamma)}{\partial\gamma}ds$$

$= -\int_0^\infty U''(s - x_1)\frac{\partial F(s;\gamma)}{\partial\gamma}ds$. Integration by parts, along with Leibniz' Rule once more yields then:

[18] See Peter A. Diamond and Joseph. E. Stiglitz (1974), Increases in Risk and in Risk Aversion, *Journal of Economic Theory*, 8 (3), 337 – 360. See also chapter 2 of a book by Jean-Jacques Laffont (1989), *The Economics of Uncertainty and Information*, MIT-Press.

$$-\int_0^\infty U''(s-x_1)\frac{\partial F(s;\gamma)}{\partial\gamma}ds = -\int_0^\infty U''(s-x_1)\left\{\frac{d}{ds}\int_0^s \frac{\partial F(y;\gamma)}{\partial\gamma}dy\right\}ds$$

$$= -U''(s-x_1)\int_0^s \frac{\partial F(y;\gamma)}{\partial\gamma}dy\Bigg|_0^\infty + \int_0^\infty U'''(s-x_1)[\int_0^s \frac{\partial F(y;\gamma)}{\partial\gamma}dy]ds$$

$$= \int_0^\infty U'''(s-x_1)[\int_0^s \frac{\partial F(y;\gamma)}{\partial\gamma}dy]ds = \int_0^\infty U'''(s-x_1)\Psi(s;\gamma)ds$$

which is positive (negative) if $U''' > 0 \ (< 0)$ and if this third derivate is uniformly signed for all s. (Remember that SSD implies that $\Psi(s;\gamma) := \int_0^s \frac{\partial F(y;\gamma)}{\partial\gamma}dy \geq 0$.)

We then have the final result:

Result #3

We have, $-sign\,\frac{dx_1(\gamma)}{d\gamma} = sign\ \beta\int_0^\infty U'(s-x_1)\frac{\partial^2 F(s;\gamma)}{\partial s\partial\gamma}ds = sign\ \beta\int_0^\infty U'''(s-x_1)\Psi(s;\gamma)ds$. If $U''' > 0$ or if marginal utility of second-period consumption, U', is convex, then a mean-preserving increase in risk will reduce first-period extraction or increase first-period resource saving. On the other hand, if $U''' < 0$, or if U' itself is concave, then a mean-preserving increase in spread, will increase first-period extraction or reduce first-period resource saving.

Before we provide some explanation of this result, let me just point at the following: In the literature on risk and uncertainty, one defines *prudence* if more risk (in the sense of MPS), leads to higher first-period saving. A result from Gollier (2001)[19] says that: *An agent is prudent if and only if the marginal utility of future consumption is convex.*

So what is the intuition behind this result? We know, on using Jensen's inequality, that if U is concave then $U(ES) > EU(S)$, with S as the stochastic variable. (If U is convex, then we have $U(ES) < EU(S)$.)

[19] Se chapter 16 in Christian Gollier (2001), *The Economics of Risk and Time*, MIT-Press.

According to Result #2 above we know that a mean-preserving increase in risk will reduce expected utility if the utility is concave, as has been assumed. On the other hand, expected utility will increase with more risk if U is convex.
Using this we see: If marginal utility U' is convex, i.e. $U''' > 0$, then an increase in risk (in the sense of MPS) will increase the value of EU'. If we look at the first-order condition for our original problem written as, $U'(x_1) = \beta E_S U'(S - x_1)$, we know that more risk (in the sense of MPS) will increase the RHS of this condition. In order to retain equality, the LHS must increase as well. With declining marginal utility (concavity of the utility function), this is equivalent to say that x_1 must go down, as shown above. Hence, the resource saving is increased. In a similar way it is easy to demonstrate that if U' is concave, then more risk, will reduce $E_S U'(S - x_1)$. To retain the first-order condition, $U'(x_1)$ must go down, or x_1 must increase because $U'' < 0$.[20]

[20] Applying the results from this section should enable you to read – and understand – the nice 1978-article by Michael Hoel.