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ECON5160: On portfolio separation

This paper concerns the portfolio optimization problem for a small agent in a frictionless continuous-time market where prices are geometric independent increment processes with elliptically distributed increments. *Portfolio separation* is the property that a (large) market can be reduced to a few indices («funds») without the agents being worse off. The simplest case, two-fund separation, reduces the number of funds to 2, and the optimization to the (one-dimensional!) allocation between those. We call it have *monetary* separation if there is a monetary account which can be taken as one of the funds; i.e., two-fund monetary separation is the property that one can find an index and allocate the investment between this index and safe money.

Whether the separation property holds, is a matter of the *agents' preferences* and/or the *probability distribution of the returns vector*. Starting from Tobin [13], generalizations have to a great extent gone either in the direction of characterizing the preferences which admit separation regardless of distribution (Cass and Stiglitz [1], discrete time, if the utility function is smooth) or a characterization in terms of distributions (Ross [11]provides necessary and sufficient conditions, and Owen and Rabinovitch [8] and Chamberlain [2] show that elliptically distributed returns satisfy these, by *mean–variance trade-off*). However, portfolio separation can also stem from other approaches, like CAPM (which of course has been scrutinized extensively over the years), or much later through the application of risk measures, as pointed out by this author [3] and recently by De Giorgi et al. [5]. The theory of portfolio separation is still a topic for research, see e.g. Schachermayer et al. [12], much in the spirit of [1] although using modern probabilistic tools.

The usual approach in the mean–variance setting, is to minimize variance given mean – which assumes risk aversion. Under risk aversion, two fund separation in the continuous-time lognormal model was obtained by Merton [7] by means of dynamic programming. The assumption of risk aversion can be dropped: Instead of minimizing volatility given mean, Khanna and Kulldorff [6] choose to maximize mean given the volatility, and are by remarkably simple methods able to remove the risk aversion and completeness assumption and also allow for «no short sale» constraints on a subset of the portfolios, as well as incomplete markets. They do however assume the existence of a risk-free investment opportunity, an assumption dropped herein. Furthermore, this paper shows that the Owen–Rabinovich and Chamberlain results carry over to the continuous-time setting, assuming that the distributions have a continuous-time analogue (i.e., that they are infinitely divisible).

This paper removes a few of the assumptions of [6], simplifying the approach, but admitting portfolio restrictions, including cases without risk-free investment opportunity, as well as a wider class of distributions. Elliptical (a.k.a. «elliptically contoured») driving noise distributions (cf. [8] and [2]) will be covered – here, some generality will be left out by assuming that the numéraire is a constant (i.e., the discounting with a numéraire is assumed done already). In order to stick to the continuous-time setting, we shall also assume infinite divisibility, merely noting that the arguments can be copied almost verbatim for discrete time if desired, as is well known in the literature.

0.1 Assumptions and notation. Throughout the paper, boldface italics symbols denote vectors, and their transposes by the $^\top$ superscript. $\mathbf{R}$ will denote the reals. We will also frequently suppress time-dependence of the parameters and choice variables. All stochastic time-differentials («d$\boldsymbol{Z}$» etc.) will be assumed to be of the Itô (non-anticipative) type. △

[1] Research manuscript shortened down for the purpose of the course

1 The wealth dynamics in traded and non-traded markets

Let us first consider the usual model for a stock market with n risky investment opportunities $\{S_i\}$ each satisfying an Itô (non-anticipative) stochastic differential equation with driving noise $\boldsymbol{Z}$ (see however Remark 1.1):

$$\mathrm{d}S_i(t) = S_i(t)\big[\mu_i\,\mathrm{d}t + \boldsymbol{\sigma}_i{}^\top \mathrm{d}\boldsymbol{Z}(t)\big] \tag{1}$$

where the coefficients μ_i and $\boldsymbol{\sigma}_i$ are deterministic functions. We shall later treat both the case where a safe investment opportunity exists and the case where it does not; assume for the moment that the model's numéraire exists as an investment opportunity with price $S_0(t) = 1$ for all t, i.e. we assume that we work with discounted figures. We remark that discounting accounts for the common additive risk component found elsewhere in the literature, e.g. the y in [11, Theorems 1 to 3].

We form a portfolio from the investments: If at time t one has $\nu_i(t)$ units of investment i, then the market value of the portfolio is $\sum \nu_i(t)S_i(t)$. In discrete time, the self financing condition says that the change in the market value of the portfolio should come from changes in the prices, so that change in wealth should be $\sum \nu_i(t)\big(S_i(t+h) - S_i(t)\big)$ for $h > 0$. The analogous requirement can be taken as a definition of a self-financing portfolio in our continuous-time setting, i.e. wealth fluctuating as $\sum \nu_i(t)\,\mathrm{d}S_i(t)$; however, from our wealth we will also deduct an amount for consumption. Letting $C(t)$ be the (discounted) cumulative net consumption up to time t and assume that the portfolio is self-financing apart from the consumption, we have that wealth at time t is $X(t) = \sum \nu_i S_i(t)$, and developing according to

$$\begin{aligned}\mathrm{d}X(t) &= \sum_{i=0}^{n} \nu_i(t)\,\mathrm{d}S_i(t) - \mathrm{d}C(t)\\ &= \sum_{i=1}^{n} \nu_i(t)S_i(t)\big[\mu_i\,\mathrm{d}t + \boldsymbol{\sigma}_i{}^\top\,\mathrm{d}\boldsymbol{Z}(t)\big] - \mathrm{d}C(t)\end{aligned}$$

(sum from $i = 1$ since we consider discounted figures), so that

$$\mathrm{d}X(t) = \boldsymbol{u}^\top\big[\boldsymbol{\mu}\,\mathrm{d}t + \Sigma\,\mathrm{d}\boldsymbol{Z}(t)\big] - \mathrm{d}C(t) \tag{2}$$

where Σ has rows $\{\sigma_i{}^\top\}$ and our control $\boldsymbol{u} = \boldsymbol{u}(t)$ measures the value held in the risky investment at time t (see Remark 1.1 for the issue of jumps to zero); the amount invested safely will be $X - \boldsymbol{u}^\top\mathbf{1}$.

As mentioned at the beginning of this section, the above is the canonical model for a (frictionless) *traded market*. For an *insurance premiums and claims model*, on the other hand, a simple model is as follows: For each distinct policy or pool i of which the insurer is responsible for a fraction ξ_i, the insurer will get a discounted premium income flow rate $\xi_i\mu_i$ but be exposed to losses; we assume that discounted cumulative losses evolve according as $-\xi_i\boldsymbol{\sigma}_i{}^\top\,\mathrm{d}\boldsymbol{Z}(t)$. In this case we do not need the self financing concept; we simply assume that at each time t the insurer can choose ξ_i freely, i.e. what contracts to undertake and how much of each. Then we arrive at (2) directly (no self financing concept invoked) but with the interpretation that $\mathrm{d}C$ represents dividend flow from the insurance company and that $u_i = \xi_i$.

We see that both the «traded market» model and the «insurance» model lead to dynamics of the form (2). However, a stock and an insurance contract are different in some aspects. By limited liability, a stock price model should not admit negative values, implying that $\sigma_{ij}\,\mathrm{d}Z_j$ should almost surely not jump by a factor less than -1. An insurance contract, on the other hand, should with positive probability lead to losses for the insurer – and there is no reason to assume this lower bound on the jumps. We will take (2) as our model assumption. Due to previous feedback to the author, it is expressedly emphasized that limited liability is not *assumed* – rather, it does for some of the distributions treated herein, follow from the model.

1.1 Remark. Taking (2) as assumption imposes some interpretations for the event where prices hit zero. The basic model (1) suggests that from then on, this investment opportunity will be dead and gone. However, (2) is only compatible with the opportunity being instantly reborn with same dynamics and nonzero price (for example, modelling the case of catastrophe bonds which are rendered worthless at a stopping time, but where new bonds are immediately issued). The alternative interpretation – that the number of investment opportunities changes as price hits zero – is shown by this author [4] to disrupt the portfolio separation property in some cases. Notice however that prices will not hit zero unless there is a point probability for a jump by a factor of -1. △

2 Stochastic dominance and mean-variance efficiency in the Gaussian case

Consider the model (2) and assume $\boldsymbol{Z}$ is a standard Brownian motion; write $M := \Sigma R \Sigma^\top$ for the volatility matrix. The concept of *mean-variance efficiency* seems to first to have occured with de Finetti in 1940, see [9]; in its well-known conventional form, it focuses on minimizing variance for given mean. For our purposes, it turns out more convenient to maximize mean for given level of variance, even though it may sometimes require an additional fund (take for example the case where all risky investment opportunities have zero-mean returns; then all risk averse agent will choose only the safe investment, but risk seekers may not). Drift maximization is the approach of [6], and we shall use a similar argument (even in subsequent sections, where distributions may fails integrability).

Let us for the moment proceed intuitively. Up to now, we have not specified whether the agent is free to choose $\boldsymbol{u}$. Let us assume that $\boldsymbol{u}$ is required to be predictable and for each t belong to some given closed set U. The mean-variance optimization problem – or rather, «drift-volatility» in a continuous time model – now becomes:

$$\max_{\boldsymbol{u}\in U} \boldsymbol{u}^\top \boldsymbol{\mu} \quad \text{subject to } \boldsymbol{u}^\top M \boldsymbol{u} = Q^2. \tag{3}$$

Now assume that we are given some strategy $(C, \boldsymbol{u}) = \{(C(t), \boldsymbol{u}(t))\}_t$. Let for each t $\boldsymbol{u}^*(t)$ solve (3) for $Q^2 = \boldsymbol{u}^\top(t) M(t) \boldsymbol{u}(t)$. By construction, this will yield higher drift at the same volatility. Now define C^* by

$$\mathrm{d}C^*(t) = \mathrm{d}C(t) + (\boldsymbol{u}^*(t) - \boldsymbol{u}(t))^\top \boldsymbol{\mu}(t)\,\mathrm{d}t$$

and assume $C^* \in U$. Compare the two strategies $(C, \boldsymbol{u})$ and $(C^*, \boldsymbol{u}^*)$. By definition, the (Itô) stochastic integral is a limit of sums

$$\sum_d \boldsymbol{u}(t_d)^\top \big(\boldsymbol{\mu}[t_{d+1} - t_d] + \Sigma(t_d)[\boldsymbol{Z}(t_{d+1}) - \boldsymbol{Z}(t_d)]\big). \tag{4a}$$

Each (d-)term has a conditional univariate Gaussian distribution with thus uniquely determined by mean and variance. Passing to the limit, we therefore have that there exists a real-valued standard Brownian motion Z such that

$$\boldsymbol{u}^{*\top} \Sigma\,\mathrm{d}\boldsymbol{Z} \overset{\mathfrak{D}}{=} Q\,\mathrm{d}Z \overset{\mathfrak{D}}{=} \boldsymbol{u}^\top \Sigma\,\mathrm{d}\boldsymbol{Z} \tag{4b}$$

where «$\overset{\mathfrak{D}}{=}$» denotes coincidence in probability law. Now consumption will possibly depend on past wealth, so let us for the moment introduce the notation $C(t) = F(X_{s\leq t})$ to express this dependence. Note in particular that F corresponds to the given $(C, \boldsymbol{u})$ strategy, not the $(C^*, \boldsymbol{u}^*)$ strategy. Corresponding to the strategy $(C^*, \boldsymbol{u}^*)$, consider the following fictious modification of the wealth-consumption pair (X^*, C^*): we still withdraw $\mathrm{d}C^*$ from our wealth, but only $\mathrm{d}F(X^*_{s\leq t}) = \mathrm{d}C^* - (\boldsymbol{u}^* - \boldsymbol{u})^\top \boldsymbol{\mu}\,\mathrm{d}t$ is actually consumed, while the remaining (nonnegative!) portion $(\boldsymbol{u}^* - \boldsymbol{u})^\top \boldsymbol{\mu}\,\mathrm{d}t$

is thrown away. Thus the actual wealth-consumption pair (X^*, C^*) is replaced by $(X^*, F(X^*))$, which yields:

$$\begin{aligned}(\mathrm{d}X^*(t), \mathrm{d}F(X^*_{s\le t})) &= (\boldsymbol{u}^{*\top}\boldsymbol{\mu}\,\mathrm{d}t + \boldsymbol{u}^{*\top}\Sigma\,\mathrm{d}\boldsymbol{Z} - \mathrm{d}C^*, \mathrm{d}F(X^*_{s\le t}))\\ &= (\boldsymbol{u}^{\top}\boldsymbol{\mu}\,\mathrm{d}t + \boldsymbol{u}^{*\top}\Sigma\,\mathrm{d}\boldsymbol{Z} - \mathrm{d}F(X^*_{s\le t}), \mathrm{d}F(X^*_{s\le t}))\\ &\overset{\mathfrak{D}}{=} (\boldsymbol{u}^{\top}\boldsymbol{\mu}\,\mathrm{d}t + \boldsymbol{u}^{\top}\Sigma\,\mathrm{d}\boldsymbol{Z} - \mathrm{d}F(X_{s\le t}), \mathrm{d}F(X_{s\le t}))\end{aligned} \tag{4c}$$

if $(X^*, F(X^*)) \overset{\mathfrak{D}}{=} (X, F(X))$ up to time t, so then $(X^*, F(X^*)) \overset{\mathfrak{D}}{=} (X, F(X))$ up to $t + \mathrm{d}t$, hence everywhere. More rigorously, equivalence in law up to time t_d for the Riemann sum approximation implies equivalence in law up to time t_{d+1} and thus everywhere; then pass to the limit.

This stochastic dominance argument shows that the strategy $(C^*, \boldsymbol{u}^*)$ is just as «good» as $(C, \boldsymbol{u})$ (heuristically, at the moment, assuming free disposal – in the next section, preferences will be assumed ordered). Now if all agents use such a $\boldsymbol{u}^*$, we will have portfolio separation: Let us see what happens if $\boldsymbol{u}$ is restricted to a cone U with vertex at the origin (and the market is arbitrage-free.) Let the vector $\boldsymbol{f}$ solve problem (3) for the particular value $Q = 1$. Consider an arbitrary strategy $(C, \boldsymbol{u})$ with $\boldsymbol{u}(t)$ taking values in $\boldsymbol{u}$; then the strategy $(C^*, \boldsymbol{u}^*)$ defined by

$$\boldsymbol{u}^*(t) = \left[\boldsymbol{u}(t)^\top M(t)\boldsymbol{u}(t)\right]^{1/2} \boldsymbol{f} \tag{5a}$$

$$\mathrm{d}C^* = \mathrm{d}C(t) + (\boldsymbol{u}^*(t) - \boldsymbol{u}(t))^\top \boldsymbol{\mu}(t)\,\mathrm{d}t \tag{5b}$$

dominates $(C, \boldsymbol{u})$ in the sense of (4), and by the cone assumption, $\boldsymbol{u}^*$ does take values in U. But in (5a) $\boldsymbol{f}$ is a fixed vector common to all agent – which is (monetary) two fund separation, the other fund being the safe opportunity.

3 The assumptions needed

We shall see what assumptions we need for to make the argument of the previous section entirely rigorous. Let us note that we do not assume existence of an optimal strategy. In return, we have to stick to a slightly weaker concept of portfolio separation than usual, but in the presence of an optimal strategy, it coincides with the usual definition:

3.1 Definition (m fund separation)**.** Fix U. We shall say that we have m *fund separation* (relative to U) if there exist m funds independent of wealth such that for each admissible (consumption, portfolio) pair there is one which is (weakly) preferred and whose portfolio consists of the m funds. △

We note that we will frequently consider U's that not all agents can satisfy. For example, an agent with net debt cannot possibly own a positive position in all investments (including the safe if it exists.) Therefore separation must be considered only within the class of agents for which the U applies. See however the last part of Section 4.

3.2 Assumptions.

(a) Non-anticipativity. The strategies should be predictable and admit unique solution to (2).

(b) Weak greed. Preferences are assumed to be compatible with first-order stochastic dominance; i.e., they should form a partial ordering on the (wealth, consumption) pairs such that (X^*, C^*) is (possibly weakly) preferred to (X, C) if

$$(X^*, C^*) \overset{\mathfrak{D}}{=} (X, C + \int^{\cdot} c\,\mathrm{d}t) \quad \text{for some predictable } c \ge 0.$$

(c) Cumulative consumption must not covariate with $\boldsymbol{Z}$, i.e. in terms of Itô differentials we require

$$\mathrm{d}C\,\mathrm{d}\boldsymbol{Z} = \boldsymbol{0}.$$

(d) Existence of some admissible strategy.

(e) Probability distributions must permit the construction (4).

△

3.3 Remark. We make a few remarks on the assumptions. For the *greed assumption* (b), we note that it covers expected increasing utility, including bonus schemes with call option or binary option structure; also it covers risk measured by quantiles (value-at-risk or ruin probability), and a wide range of lexicographical orderings. We note that we may *restrict the consumption processes* even further than merely (c) – e.g. by imposing lower boundedness or nonnegativity on C or its growth rate (time-derivative) – but we must for each admissible C admit the $C + \int^{\cdot} c\,\mathrm{d}t$ construction in (b) (for given law of the wealth process). The restriction (c) is still a generalization of the usual assumption that consumption rates are positive; we do allow for e.g. non-financial income, but anything which covariates with the market, must be part of the market model. Regarding existence: We obviously need an admissible strategy, but we shall not assume any compactness needed for optimum to exist. Indeed, in the Gaussian case (where prices are continuous), if an increasing utility function is concave–convex–concave, then on the convex part one will want to inflate to infinite volatility until hitting the convex hull. As for the *probability laws*, we used in Section 2 the property that the zero-mean normal distributions constitute a one parameter family which is closed not only under convolution, but under suitable linear combinations as well. The *elliptical* (also known as *elliptically contoured*) distributions form a suitable generalization in the case where we allow for arbitrary linear combinations, i.e. short sale permitted. If we only work with *positive* linear combinations, then the family of α-stable distributions (elliptical or not) does permit the construction. Using jointly α-stable distributions we can repeat the coinciding law argument (4) in an entirely rigorous manner subject to skewness and index of stability. Under the fairly mild assumptions 3.2, every agent is a drift-volatility optimizer in the Gaussian case: in the non-Gaussian case, there will essentially only be a different volatility measure, except for the oddball skew Cauchy distribution. △

For completeness, we give a few basic properties of the distribution classes in question.

4 The elliptical case

Following [8, Definition (a)], we recall that an *elliptical*, also known as an *elliptically contoured* $\mathbf{R}^n$-valued random variable $\boldsymbol{Y}$ is one for which the characteristic function separates into

$$\mathsf{E}[\exp(i\boldsymbol{u}^\top \boldsymbol{Y})] = \exp(i\boldsymbol{u}^\top \boldsymbol{d})\ \Psi(\boldsymbol{u}^\top M \boldsymbol{u}) \tag{6}$$

where $\boldsymbol{d}$ is a vector and M is a positive definite matrix. If Σ is invertible, and $\boldsymbol{Y}$ is an increment of $\boldsymbol{Z}$ satisfying this definition with $\mathbf{0}$ and R for $\boldsymbol{d}$, M respectively, then we will have $\boldsymbol{Y}$ elliptical: $\mathsf{E}[\exp(i\boldsymbol{u}^\top \boldsymbol{Y})] = \Psi(\boldsymbol{u}^\top M \boldsymbol{u})$, with, again, $M = \Sigma R \Sigma^\top$. Since we have drift in the model, we can incorporate $\boldsymbol{d}$ in this drift term, and assume it to be zero. We say that the distribution is *elliptical about the origin.*

From the definition of ellipticity it follows that all linear combinations – which translates to instantaneous returns less drift, when $\boldsymbol{u}$ being interpreted as portfolio – are univariate elliptical, characterized by the quadratic form $\boldsymbol{u}^\top M \boldsymbol{u}$. In addition we have drift, which in our case becomes $\boldsymbol{u}^\top \boldsymbol{\mu}$. For the construction (4), it therefore suffices that the portfolio $\boldsymbol{u}$ maximizes drift $\boldsymbol{u}^\top \boldsymbol{\mu}$ given $\boldsymbol{u}^\top M \boldsymbol{u}$: Just like in (4a), the Itô integral is now the limit in probability of sums of terms of the form

$$\boldsymbol{u}^\top(t_d^-)[\boldsymbol{Z}(t_{d+1}) - \boldsymbol{Z}(t_d)] \tag{7}$$

which is now a drift term plus a real-valued random variable elliptically distributed about the origin. For given value of the quadratic form, then (4b) and (4c) follow with the modification that Z now is a Lévy motion with (necessarily infinitely divisible and) elliptic increments.

For the result, we notice that we can restrict the portfolio weights to a quadrant (e.g. forbidding short sale), cone, double cone, or more generally a family of half-lines from the origin:

4.1 Theorem (Two fund separation)**.** *Assume the dynamics to follow* (2)*, with the increments of* $\boldsymbol{Z}$ *being elliptical about the origin and infinitely divisible. Assume precisely one of the following two restrictions* U *on the portfolio vectors* $\boldsymbol{u}$*:*

- U *is a family of half-lines from the origin, or*
- $U = \{\boldsymbol{u};\ \boldsymbol{u}^\top \mathbf{1} = X\}$ *(i.e., no safe investment opportunity).*

Then we have two fund separation if there is no arbitrage.

Proof. The first case follows as in section 2: Let $\boldsymbol{f}$ be the vector which solves the problem of maximizing drift subject to $\boldsymbol{u}^\top M\boldsymbol{u} = 1$. Then for any $Q \geq 0$, $Q\boldsymbol{f} \in U$ and maximizes drift subject to $\boldsymbol{u}^\top M\boldsymbol{u} = Q^2$. Now for any given strategy $(C, \boldsymbol{u})$, observe that the pair $(C^*, \boldsymbol{u}^*)$ constructed by (5) is preferred by Assumption 3.2 (b). As for the case of no risk-free investment opportunity, note that this corresponds to the constraint $\boldsymbol{u}^\top \mathbf{1} = X$. Consider the problem

$$\max \boldsymbol{u}^\top \boldsymbol{\mu} \quad \text{subject to} \qquad \boldsymbol{u}^\top M\boldsymbol{u} = Q^2, \qquad \boldsymbol{u}^\top \mathbf{1} = X. \tag{8}$$

Except the case where the two constraints form a singleton (the point $\frac{X}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1}$, which will be covered by the formula to follow), we have the constraint qualification satisfied, and the Lagrange first-order condition reads

$$\boldsymbol{\mu} = 2\Lambda M\boldsymbol{u} + \lambda\mathbf{1} \tag{9}$$

Since M is invertible, we have $\boldsymbol{u}$ separating into a linear combination of the two funds $M^{-1}\boldsymbol{\mu}$ and $M^{-1}\mathbf{1}$, provided that $\Lambda \neq 0$. Should Λ vanish, then $\boldsymbol{\mu}$ must be a multiple of $\mathbf{1}$, in which case the maximization degenerates. Then choose $\frac{X}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1}$ and any nonzero fund orthogonal to $\mathbf{1}$. Then repeat the construction (5). □

4.2 Remark. The $\frac{X}{\mathbf{1}^\top M^{-1}\mathbf{1}} M^{-1}\mathbf{1}$ vector is the «minimum variance portfolio» in the classical case. The «nonzero fund orthogonal to $\mathbf{1}$» is then pure volatility, in the sense that it does not contribute to expected return, and is usually not stated in the literature. It is needed in our case, since we have not assumed risk aversion. For similar reasons, we cannot rule out the need for an additional fund if there is an arbitrage opportunity (corresponding to semi-definite M): the arbitrage portfolio funded by borrowing, both of which will then be scaled to infinity, and then a fund in case volatility is desired. △

We can cover no-borrowing constraints, or even different lending and borrowing interest rates, at the cost of additional funds. Let the «$\lessgtr$» symbol denote either $\leq$ or $=$, and assume that we are given restrictions of the form

$$\boldsymbol{u}^\top \boldsymbol{a}_j \lessgtr z_j, \qquad j = 1, \ldots, k \tag{10}$$

where the $\boldsymbol{a}_j$ are common to all agents, but the z_j may be individual. The Lagrangian associated to the static optimization problem now becomes

$$L = \boldsymbol{u}^\top(\boldsymbol{\mu} - \sum_j \lambda_j \boldsymbol{a}_j) - \Lambda \boldsymbol{u}^\top M\boldsymbol{u} \tag{11}$$

If the constraint qualification fails, then analogously to Theorem 4.1 we must be at the singleton where the ellipsoid is tangent to the convex polyhedron defined by the linear constraints. This case will be a limiting case in Q and does not require special treatment. If the Lagrange conditions hold, we have

$$\Lambda \boldsymbol{u} = M^{-1}(\boldsymbol{\mu} - \sum_j \lambda_j \boldsymbol{a}_j), \tag{12}$$

where the right-hand side is spanned by at most $k+1$ vectors, i.e. at most $k+2$ funds. What remains is the case $\Lambda = 0$, for which the «optimal» choice must be reserved to fulfilling admissibility, with drift being constant. In this case, observe that the vector $M^{-1}\mu$ (which is one vector in the expansion (12), and it is redundant due to linear dependency) is minimizing the quadratic for given drift. An additional fund orthogonal to $\boldsymbol{\mu}$ then suffices, and we still only have $k+2$ funds. We have shown:

4.3 Theorem ($k+2$ fund separation)**.** *Assume the dynamics to follow* (2)*, with* $\boldsymbol{Z}$ *being elliptical about the origin, and the portfolio constrained to satisfy* (10)*. Then we have* $k+2$ *fund separation (*$k+1$ *funds if* (10) *forbids a risk-free opportunity).*

In order to accommodate different rates for lending and borrowing, corresponding to $\boldsymbol{u}^\top \mathbf{1}$ being $\leq$ resp. $\geq$ than X, we simply assume an interest premium $r(K)$ on $K = \boldsymbol{u}^\top \boldsymbol{a}_0$ (K may be individual, and so may even $r(k)$) – the obvious choice is $\boldsymbol{a}_0 = \mathbf{1}$, but mathematically that is just a special case. The result is

4.4 Theorem (Different interest rates require one more fund)**.** *Consider the setting of Theorem 4.3, but with individual interest rates, possibly depending on* $\boldsymbol{a}_0$*. We then have* $k+3$ *fund separation (in particular,* 3 *fund separation if there are no constraints in* (10)*).*

Proof. For each K, the agent will want to maximize drift $\boldsymbol{u}^\top(\boldsymbol{\mu} - r(K)\boldsymbol{a}_0) = -Kr(K) + \boldsymbol{u}^\top\boldsymbol{\mu}$, subject to (10) augmented with $\boldsymbol{u}^\top \boldsymbol{a}_0 = K$, which gives just another fund with K-dependent Lagrange multiplier. □

4.5 Remark. Note that the extra fund is needed also if the interest rate is fixed per agent but varies between them. Note also that the generality covers leverage constraints applying to some of the agents (e.g., if only some of the agents are allowed to borrow). △

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