UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Home assignment: **ECON5160/9160 – Stochastic Modeling and Analysis**, spring 2011.

Handed out: May 31, 2011 at 09:00

To be delivered by: June 3, 2011 at 14:00

Place of delivery: Department office, 12th floor

Further instructions:

- The questions are in English, but you can give your answers in English, Norwegian, Swedish or Danish.
- The home assignment will be marked and the scale for the mark will be **A** (best) through **E** for passes, and **F** for fail.
- After completion, please hand in **2 – two –** copies of your paper to the address given above. The papers must not bear your name, but the individual examination number, which you can find in your studentweb.
- In addition, you must fill in the **declaration on compliance with the rules**, available at the course website. Your answer must fill the formal requirements, found at http://www.sv.uio.no/studier/ressurser/kildebruk/ (Norwegian) or http://www.sv.uio.no/english/studies/resources/sources-and-references/ (English).
- It is of importance that your paper is delivered by the deadline (see above). Papers delivered after the deadline, **will not be assessed nor marked**.*)
- All papers must be delivered to the place given above. You must not deliver your paper to the course teacher or send it by e-mail. If you want to hand in your paper **before** the deadline, please contact the department office on 12th floor.

*) The rules for illness during exam also applies for the home exam. Please see http://www.sv.uio.no/english/studies/admin/exams/postponed-exam/index.html for further details.

University of Oslo / Department of Economics / 2011–05–31

ECON5160/9160 (Stochastic Modeling and Analysis): Exam spring 2011, May 31st – June 3rd

- There are 3 pages of problems, in addition to the cover note (1 page) and declaration form (separate PDF document).
- All answers must be justified. Show your math; reference to e.g. computer calculations, is not sufficient justification (but you are allowed to use such tools for your own verification).
- For further information, refer to the cover note.

Problem 1 Let $d \in [0,1]$ and $c = (1-d)/4$. A Markov chain has transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} d & c & c & c & c & 0 & 0 \\ 0 & .1 & .2 & .3 & .4 & 0 & 0 \\ 0 & .2 & .3 & .4 & .1 & 0 & 0 \\ 0 & .3 & .4 & .1 & .2 & 0 & 0 \\ 0 & .4 & .1 & .2 & .3 & 0 & 0 \\ 0 & c & c & c & c & 0 & d \\ 0 & c & c & c & c & d & 0 \end{bmatrix} \end{array}$$

(a). Determine, for each value of $d \in [0,1]$ the positive recurrent state(s) (if any), and the null recurrent states (if any).

(b). Determine, for each value of $d \in [0,1]$, the communication classes.

(c). For what – if any – value(s) of d will the Markov chain be regular?

(d). Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_6) = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0)$. For what – if any – value(s) of d will $\boldsymbol{\pi}$ be

i) a stationary distribution?

ii) a limiting distribution?

(e). Assume that the Markov chain starts in state 6. Determine, for each value of $d \in [0,1]$ for which state 6 is transient, the expected time until the Markov chain leaves the communication class state 6 belongs to.

Problem 2 Throughout this problem, Y will be a birth/death-process with state space $\{0, 1, 2\}$. The intensity of an upward jump from state $i \in \{0, 1\}$ will be $2 - i$. The intensity of a downward jump from state $i \in \{1, 2\}$ will be $\frac{1}{2}i^2$.

(a). Write down

i) the generator,

ii) the Kolmogorov backward equation (i.e. equation system)

and

iii) the Kolmogorov forward equation (i.e. equation system)

associated to Y.

(b). In the long run, what fraction of the time will Y spend in state 1?

Problem 3 Throughout this problem, $M(t)$ will be a martingale in continuous time, starting at $M(0) = 0$, τ will be first time for which $|M| \geq 1$, and M satisfies $\mathsf{E}[(M(t))^2] < \infty$ (for all t), and $\mathsf{E}[\tau^2] < \infty$.
Let $p = \mathsf{Pr}[M(\tau) \geq 1]$.

(a). Show that $p = 1/2$ if M is continuous.

(b). In this part, assume that M has no downward jumps. Decide whether this condition

- implies that $p \geq 1/2$, or
- implies that $p \leq 1/2$, or
- is insufficient to conclude whether $p \leq 1/2$ or $\geq 1/2$.

Problem 4 Throughout this problem, let $T > 0$, $r > 0$ and $k \geq 1$ be given constants. Consider for time $t \in [0, T]$ a frictionless arbitrage-free market consisting of one investment opportunity whose price at time t is deterministic and equal to e^{rt}, and furthermore an investment opportunity with stochastic price $S(t)$ at time t, with $S(0) = 1$. Consider three Arrow–Debreu securities with arbitrage-free prices as follows:

- security #1 pays 1 if $S(T) \geq k$ (and 0 otherwise), and has price c_1 at time 0,
- security #2 pays 1 if $S(T) \leq 1/k$ (and 0 otherwise), and has price c_2 at time 0,
- security #3 pays 1 if $k \geq S(T) \geq 1/k$ (and 0 otherwise), and has price c_3 at time 0.

(a). Find c_1 for the case when

$$S(t) = \exp\big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma B(t)\big) \qquad \text{(gBm)}$$

where $\mu \geq r$ and $\sigma \neq 0$ are given constants, and where $B(t)$ is standard Brownian motion starting at $B(0) = 0$.

(b). It is a fact that $c_1 + c_2 + c_3 = e^{-rT}$ when S is given by formula (gBm) above. However, there are other cases where there is inequality. Let $k = 1$ and find a suitable condition on the process S for which $c_1 + c_2 + c_3 \neq e^{-rT}$, and decide whether the sum is then $> e^{-rT}$ or $< e^{-rT}$. Make sure that your condition does not imply arbitrage opportunity. *For part (b), you are free to modify the set-up to a single period problem with trading only at time 0 if you prefer so.*

Problem 5 Let $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))^\top$ satisfy the stochastic differential equation

$$\mathrm{d}\mathbf{X}(t) = (\mathbf{b}(t) - \mathbf{A}\mathbf{X}(t))\mathrm{d}t + \mathbf{\Sigma}\,\mathrm{d}\mathbf{B}(t), \qquad \mathbf{X}(0) = \mathbf{x} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{\Sigma} \in \mathbb{R}^{n\times m}$ are matrices with constant entries, $\mathbf{b}(t)$ is a deterministic function taking values in $\mathbb{R}^n$, and $\mathbf{B}$ is m-dimensional standard Brownian motion.

(a). Solve equation (1) (i.e.: write $\mathbf{X}$ as a function of $\mathbf{x}$, an integral wrt. t and an integral wrt. the Brownian motion, but do not try to evaluate all the integrals explicitely).

(b). Simplify as much as possible if $\mathbf{b}(t) = \mathbf{A}\mathbf{q}$, where $\mathbf{q}$ does not depend on t.

In parts (c) and (d), assume that $\mathbf{X}$ satisfies (1), with $\mathbf{b} = \mathbf{A}\mathbf{q}$ as in (b), and

$$\mathbf{A} = \alpha\mathbf{I} + \beta\mathbf{C}, \tag{c\&d}$$

where $\mathbf{C}$ has the property that $\mathbf{C}^2 = \gamma\mathbf{C}$. Here, α, β and γ are constant numbers.

(c). Show that

$$e^{\beta\mathbf{C}} = \mathbf{I} + \tfrac{e^{\gamma\beta}-1}{\gamma}\mathbf{C}. \tag{2}$$

(d). Use (2) to simplify the solution of (1) as much as possible. You are allowed to use equation (2) regardless of whether you managed to prove it. (*Hint:* you can use without proof the fact that $e^{\mathbf{K}+\mathbf{L}} = e^{\mathbf{K}}e^{\mathbf{L}}$ is true whenever the matrices commute, i.e. whenever $\mathbf{KL} = \mathbf{LK}$.)

Now drop the «(c&d)» condition, and consider instead for the rest of the problem, the system of stochastic differential equations for $\mathbf{X} = (X_1, X_2, X_3)^\top$:

$$\begin{aligned}
\mathrm{d}X_1(t) &= r_1(X_2(t) - X_1(t))\mathrm{d}t + \boldsymbol{\sigma}_1^\top\,\mathrm{d}\mathbf{B}(t)\\
\mathrm{d}X_2(t) &= r_2(X_3(t) - X_2(t))\mathrm{d}t + \boldsymbol{\sigma}_2^\top\,\mathrm{d}\mathbf{B}(t)\\
\mathrm{d}X_3(t) &= r_3(m - X_3(t))\mathrm{d}t + \boldsymbol{\sigma}_3^\top\,\mathrm{d}\mathbf{B}(t).
\end{aligned} \tag{3}$$

where all three r_i are > 0, and where $\mathbf{X}$ starts at $\mathbf{X}(0) = \mathbf{x}$.

(e). Write equation system (3) on the form (1), and show that when all the r_i are distinct (i.e. $r_1 \neq r_2 \neq r_3 \neq r_1$), then $\mathbf{A}$ has eigenvectors (*right* eigenvectors, not left!)

$$\mathbf{u}_1 = \begin{pmatrix}1, & 0, & 0\end{pmatrix}^\top, \quad \mathbf{u}_2 = \begin{pmatrix}1, & 1-\tfrac{r_2}{r_1}, & 0\end{pmatrix}^\top, \quad \mathbf{u}_3 = \begin{pmatrix}1, & 1-\tfrac{r_3}{r_1}, & (1-\tfrac{r_3}{r_1})(1-\tfrac{r_3}{r_2})\end{pmatrix}^\top.$$

(f). Let $\mathbf{U}$ be the matrix with columns $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. Find deterministic real-valued functions $d_1(t)$, $d_2(t)$, $d_3(t)$ such that whenever all the r_i are distinct, we have

$$e^{\mathbf{A}t} = \mathbf{U}\mathbf{\Delta}(t)\mathbf{U}^{-1}, \qquad \text{where } \mathbf{\Delta}(t) = \begin{pmatrix} d_1(t) & 0 & 0\\ 0 & d_2(t) & 0\\ 0 & 0 & d_3(t)\end{pmatrix}. \tag{4}$$

(You are not supposed to calculate $\mathbf{U}^{-1}$.)

(g). Assume that $\mathbf{\Sigma}$ is the identity matrix. Use your diagonalisation (4) to simplify the solution $\mathbf{X}$ of (3). Your formula should not include $\mathbf{A}$ explicitely. (Again, you are not supposed to calculate $\mathbf{U}^{-1}$.)

(h). Again, consider equation system (3) written on the form (1), but assume now that $r_1 = r_2 = r_3$. Is the matrix $\mathbf{A}$ now diagonalisable?