University of Oslo / Department of Economics / NCF 2011–05–26

ECON5160: Theory overview

This note is intended to give a rough list of the most essential *theory* covered.

TK/PK I. The basic concepts of probability are preliminaries to this course. For exam purposes, they are tools to be utilized in solving the problems.

TK/PK II. From ch. II, the random sum material (II.3) serves as an introduction to e.g. compound Poisson processes later on.
The *martingale* theory (II.5) was covered later in the course. You should be able to

- Use the two results due to Doob, namely the maximal inequality and the optional sampling theorem, both in continuous and discrete time. (The treatment of optional sampling goes beyond the books.)
- In continuous time: Recognize a stochastic integral wrt. a martingale as a martingale (modulo the integrability condition). See the «solutions» note.

TK/PK III. The important part of chapter III – except of course the concepts etc., which are musts – is everything about first-step analysis. This includes the star-marked section III.7 (see also the 20110211 note (clickable link), which gives an efficient linear algebra-formulated tool; you are expected to know and use the fundamental matrix concept like in problem III.7.1.[1])

TK/PK IV. There are a lot of important concepts and tools in chapter 4. You are for example expected to be able to

- identify a *regular* transition matrix
- identify *stationary distributions* by way of *eigenvector methods*
- identify *limiting distributions* both by identifying a stationary distribution as limiting, and by way of the basic limit theorem
- classify states
- identify *aperiodic* Markov chains (and also periodic as «not aperiodic», but the period of a Markov chain is not covered in the lectures)
- for not too complicated examples, identify positive recurrent, null recurrent and transient classes as done in section IV.5.

Some of you might have read the 2009 version of this note, where problems IV.1.3 and IV.4.4 were recommended, as well as problem IV.3.2. The recommendation still stands, but this year's version of the course did go somewhat deeper on the matter.

[1] For an idea on what you should understand of infinite-dimensional matrices in this context, see the «solutions» note for this problem, where the infinite-dimensional case can be solved inductively

TK/PK V. Poisson processes etc.

There are certainly a few probability calculations needed in this chapter and subsequently – the main objective of this chapter is to give preliminaries to section VI and the stochastic calculus part.

Nevertheless, you are expected to be able to treat compound Poisson processes as random sums, to perform calculations by conditioning on the Poisson jumps etc. From the problems given, I would certainly hold problem V.6.10 to be too hard.

Skip: V.4.1, V.4.2

TK/PK VI.

- You must, of course, know birth, death and birth/death-processes, their limiting behaviours in terms of eigenvectors, and processes with absorbing state(s) and their properties in terms of «infinitesimal first-step analysis» of even last-step analysis (section 5).
  For continuous-time discrete state-space Markov chains (not necessarily a birth/death-process), you must know the crucial concept of infinitesimal generator (as in section 6) – and again, stationary distributions in terms of eigenvectors.
- Unlike the 2009 version of the course, the Kolmogorov forward and backward equations were covered in this year's version. You should be able to calculate limiting behaviour of birth/death processes.
- A detail: For intensity-driven processes, the intensity itself is the fundamental starting point. If a problem is given in terms of an «intensity», you are not supposed to do the derivation by discrete time and «$o(h)$» – rather, take the intensity as-is.
  You are expected to be able to manipulate intensities; for example, if the jump intensity at state n is $\lambda(n)$, and there is an independent random variable governing the new state at the jump, say $\mathsf{Pr}[A]$ into states $\in A$, then you are supposed to know that the intensity of a jump into A is $\lambda(n)\mathsf{Pr}[A]$ – and that for disjoint jump sets A and B, jumps into A and jumps into B can be regarded as driven by independent Poisson processes.

Skip: VI.7.

★ *Linear algebra*

You need to be able to diagonalise: Beware the distinction between left eigenvectors (the usual convention in the theory of Markov chains, e.g. $\boldsymbol{\pi}\mathbf{P} = \mathbf{P}$) and right eigenvectors (the usual convention in analysis: $\mathbf{Ax} = \lambda\mathbf{x}$.) The course has not been consistent – see in particular the «solutions» note p. 37, second bullet point.

You also need to decide whether a matrix is diagonalisable, at the level of the exposition in the «solutions» note (bottom p. 29).

You need to be able to calculate exponentials of diagonalisable matrices, and of some simpler non-diagonalisable cases.

★ *Brownian motion, geometric B.m. and Ornstein–Uhlenbeck processes*

The course has not focused on T&K's exposition of Brownian motion, rather on Brownian motion as a driving noise in stochastic differential equations (see below). Therefore, some of T&K chapter VIII may either be skipped or be considered covered better by Seierstad, Schweder and your lecture notes. In particular, we have had better use of the SDE approach to Ornstein–Uhlenbeck processes, than the T&K treatment. Compared to the 2009 version, the linear algebra part has enabled a deeper treatment of the multidimensional OU process this semester.

- Note however, from T&K: VIII.2.1 and 2.2 for some properties of the Brownian motion itself, and VIII.4 on the geometric Brownian motion (see also the mathematical finance part below).

Furthermore, the martingale property of Brownian motion is key.

★ *Stochastic calculus.* You should

- be able to apply Ito's formula
- ... and take expectations to get Dynkin's formula – and notice that this works for integrable stopping times too, by optional sampling (in particular, recognize as a martingale – modulo integrability – if there is only a $\mathrm{d}B$ integral or an integral wrt. a martingale, for example $N(t)-t\mathsf{E}[N(1)]$ where N is a Poisson process)
- ... and find the infinitesimal generator of a given process
- This 2011 version has had more treatment on stochastic calculus wrt. jump processes than previous versions. However, there is still heavier focus on the Brownian motion case.
- Only slight knowledge of the technical conditions. «Conceptual condition» are more essential (e.g.: you need to know that a $\mathrm{d}B$ integral should have a non-clairvoyant integrand, but not precisely what this means in mathematical terms).

★ *Mathematical finance*

- You should know the concepts, and you should know that some of the results from the complete-market gBm case below, may or may not generalize. In general, the lecture slides (which go beyond the books!) and the 2009 exam problem, will be the most essential reading here.
- In particular, you should know how to price derivatives written on a geometric Brownian motion by altering the dynamics, discounting and then taking ordinary expectation. To the extent that the expectations are «analytically computable» (e.g. in terms of known functions), you are expected to be able to actually do this.