ECON5160: The compulsory term paper

Formalities:

- This term paper is **compulsory**.
- This paper must be accepted in order to qualify for attending the final exam, but will **not** count on your final grade.
- To be handed in(*): Thursday April 28th 2009, at 1400 hours at the department office, 12th floor. Do not submit by e-mail unless agreed with the teachers.
- Language: English or Norwegian.
- Cooperation allowed, but the paper itself must be in your own words; similar applies for code.
- If a submitted paper is not accepted, you will get a second chance to improve it in a very short timeframe.

*) If you believe that you have good a reason for not meeting the deadline (e.g. illness), you should discuss the matter with your course teacher and seek a formal extension. Normally, an extension will be granted only when there is a good reason backed by supporting evidence (e.g. a medical certificate).

Other information:

- This document contains problems 1–4 over three pages, plus PDF attachments.
- In addition to a satisfactory total score based on the total set of compulsory problems, you are expected to make a decent attempt on each of them in order to have your paper accepted.
- For problems requiring numerical calculations, you are free to use your favourite software. Your numerical calculations must be documented:
  - If source code is available, attach a print-out
  - Otherwise, write a pseudo-code and attach
  - For some applications where the algorithm is available from a file but not in a source code format, you are welcome to discuss other means of documentation.
- We consider this term paper to be more (*much* more) work than e.g. previous Maths 2 or Maths 3 term papers.

Problem 1 Schweder problem 17, as stated. If you did problem 18, include it here.

Problem 2 This problem will consider the (attached) article of T.J. Klette and S. Kortum: «Innovating Firms and Aggregate Innovation»[1]

(a). There are given differential equations for the size distribution: (15), (16) page 1000. Write down the generator matrix, and state the key result from the curriculum which (15), (16) are a particular case of. (Hint: they are nearly the same as the state probabilities given in formula (5) p. 994, for which Appendix C refers to «standard tecniques» but sketches ad hoc the derivation . You are not supposed to copy this «first-step» derivation, but to point at the general result.)

(b). Formula (17) is derived in Appendix D. Write out this derivation in full detail.

(c). The first sentence following formula (16), says that with no entry, there will be no *limiting* distribution. Consider the case where $\eta = 0$ (which implies $\lambda = \mu$, since in general $\mu = \lambda + \eta$). Show that there cannot be any stationary distribution with $\sum_{n=1}^{\infty} M_n \in (0, \infty)$ (note: sum from 1!).
(Hint: Let $z_n = nM_n$ and show that z_n has the general solution $A + Bn$.)

Problem 3 Paul Samuelsen (1961) introduces the demand and supply curves by an example concerning wheat. Quantity is measured in bushels and price in USD. The unsmoothed demand and supply curves from his small tables are shown in the figure by solid lines. The linearized demand and supply curves are $Q = 21.3 - 2.7P$ and $Q = -2.9 + 4.5P$. The equilibrium of the linear system is $P = p = 3.36$ and $Q = q = 12.23$. The problem is to specify a stochastic dynamic model for price and quantity based on the linear system, and to simulate this process for verious specifications of the model.

(a). Assume first (against all economic reason) that demanded quantity fluctuates around the point q according to the Ornstein-Uhlenbeck process

$$\mathrm{d}Q(t) = a(q - Q(t))\mathrm{d}t + \sigma\, B(dt)$$

where B is a Brownian motion and $a,\ q,\ \sigma$ are the parameters of the model.

- Explain intuitively why $a > 0$ implies stationarity of the process.
- Use the theory (Schweder chapter 9) to find the mean and variance of $Q(t)$ when it starts at $Q(0) = 0$, and find also the stationary distribution when $a > 0$.
- What is the proportion of time the process will go negative in the long run?

The process might be simulated in various ways. We shall be satisfied with discrete time simulation, say at 1000 equally spaced time points over 100 units of time (100 years). One method is to run the process forward according to the stochastic difference equation

$$Q(t + dt) = Q(t) + a(q - Q(t))\mathrm{d}t + \sigma B(\mathrm{d}t)$$

where $B(\mathrm{d}t)$ are independent draws from $\mathcal{N}(0, \mathrm{d}t)$ and the time step $\mathrm{d}t$ is 0.1.
Two such drawn sequences $(u_1, \ldots, u_{1000})$ and $(v_1, \ldots, v_{1000})$ are attached.

[1] The plan is that the final lecture on applications will be based on this article.

(c). Do this by using the u_i sequence: $B(0.1) - B(0) = u_1, \ldots, B(100) - B(99.9) = u_{1000}$.

(d). This method is not accurate since the conditional distribution of $Q(t+dt)$ given $Q(t)$ has another distribution than $\mathcal{N}(Q(t) + a(q - Q(t))dt; \sigma^2\mathrm{d}t)$.

- What is the correct conditional distribution?
- Explain how $(Q(\mathrm{d}t), Q(2\mathrm{d}t), \ldots, Q(1000\mathrm{d}t))$ can be accurately simulated.
- Simulate one realization of the process for $\sigma = 0.1$ and one realization for $\sigma = 5$, where in both cases the stationary state is 3. Use the u_i sequence as in part (c) above. Does the realization look the same as the one from part (c)?

Prices are now brought into the picture and price and quantity, $\mathbf{X}(t) = \begin{bmatrix} Q(t) \\ P(t) \end{bmatrix}$ are simultaneously evolving according to the 2-dimensional stochastic differential equation

$$\mathrm{d}\mathbf{X}(t) = \mathbf{A}(\mathbf{X}(t) - \mathbf{x})dt + \mathbf{\Sigma}\,\mathbf{B}(\mathrm{d}t)$$

where now $\mathbf{x} = (q, p)^\top$ is the above equilibrium point, $\mathbf{A}$ and $\mathbf{\Sigma}$ are 2×2 matrices and $\mathbf{B}$ is a 2-dimensional standard Brownian motion. Let $\mathbf{A} = \alpha \begin{bmatrix} -1 & 4.5 \\ 0.36 & -1 \end{bmatrix}$ where α is a real parameter.

(e).
- Explain why this might be a dynamic and stochastic model for the linearized static system. How should α be interpreted?
- Let $\mathbf{\Sigma} = \sigma\mathbf{I}$ where $\sigma > 0$. Diagonalize $\mathbf{A}$ and find what values of α that makes the process stationary.
- What is the stationary distribution, and what is the marginal stationary distribution for Q? What must α be to have the same stationary distribution for Q as above in case $\sigma = 0.1$ and $\sigma = 5$?

(f). Solve the differential equation in terms of a $\mathrm{d}t$ integral and a $\mathrm{d}B$ integral which do not involve $\mathbf{X}$.

(g). Simulate the process over 1000 equally spaced time points over 100 time units for both coordinates. [Hint: from the diagonalization, $\mathbf{X}$ is found to be a linear combination of two independent one-dimensional processes that can be simulated as above, using both the u_i sequence and the v_i sequence.]

Problem 4 This problem is compulsory only if you did not do Schweder problem 18. Otherwise, it is voluntary (and might give nonnegative score, if you need that.)
Consider the 2-dimensional geometric Brownian motion $\mathbf{Y} = (Y_1(t), Y_2(t))^\top$ given by the stochastic differential equation

$$\mathrm{d}\mathbf{Y}(t) = \mathbf{A}\mathbf{Y}(t)\mathrm{d}t + 5\mathbf{Y}\,B(\mathrm{d}t)$$

with the initial condition $Y_1(0) = Y_2(0) = 1$. Here, B is a one-dimensional Brownian motion, and $\mathbf{A}$ is as in problem 3 with $\alpha = 1$.

(a). Simulate with the method of problem 3 (c), using the u_i sequence.

(b). Deduce a stochastic differental equation for $\mathbf{G}(t) = \exp(-\mathbf{A}t)\mathbf{Y}(t)$, and solve this equation. Does your simulation from (a) look reasonable?

(c). Use the solution from (b) to simulate $\mathbf{Y}(t)$ more accurately, using the u_i sequence. Compare the sample paths.