Microeconomics 2
Lecture notes

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Contents

Chapter 1

Adverse Selection

1.1 Introduction

As an illustration of adverse selection, consider the regulation of a public utility. The players are:

- a regulator, who is interested in the provision of a service, q, generating a gross utility $U(q)$, where $U' > 0 > U''$, and
- a firm, which faces a cost given by θq, where θ is a cost parameter ranging from $\underline{\theta}$ to $\bar{\theta}$.

The firm is paid by the regulator, who makes a transfer t to the firm; this transfer costs the regulator $(1+\lambda)t$, where $\lambda > 0$ is the shadow cost of public funds (i.e. the cost of making one unit of transfer).

Under *complete information*, the regulator's problem can be written as

$$\max_{q,t} U(q) - (1+\lambda)t$$
$$\text{s.t.} \quad t \geq \theta q.$$

where the constraint arises from the fact that the regulator must allow the firm to cover its cost of production. Since transfers are socially costly ($\lambda > 0$), they should be kept as low as possible: $t = \theta q$; plugging this into the objective function, the regulator's problem becomes

$$\max_{q} \; U(q) - (1+\lambda)\theta q.$$

The first-best solution, $q^{FB}(\theta)$, solves the first-order condition

$$U'(q) = (1+\lambda)\theta.$$

The corresponding level of transfer required is given by $t^{FB}(\theta) = \theta q^{FB}(\theta)$.

However, there is an issue: in most real-world settings, it is the firm, not the regulator, that has the best idea of the true cost. That is, the regulator does not know the true value of θ. If the regulator simply asks the firm to report θ and then enacts the outcome $(q^{FB}(\tilde{\theta}), t^{FB}(\tilde{\theta}))$ based on the firm's reported cost $\tilde{\theta}$, the firm would have an incentive to over-report, i.e. to report a cost $\tilde{\theta} > \theta$.

What can the regulator do? If the regulator were to offer a unique package (q, t), then to ensure that the contract is accepted by the firm, whatever its cost, the package should satisfy $t = \bar{\theta}q$ (so that even the highest-cost firm agrees to the package). The best such contract, i.e. the contract that maximizes $U(q) - (1 + \lambda)t = U(q) - (1 + \lambda)\bar{\theta}q$, is then $q^{FB}(\bar{\theta})$.

However, the regulator can also try to adapt the package to the cost of the firm. Indeed, using the revelation principle, the best the regulator can do is to offer a menu of options $(q(\theta), t(\theta))_{\theta \in [\underline{\theta}, \bar{\theta}]}$, satisfying the incentive-compatibility constraint:

$$\theta = \arg\max_{\tilde{\theta}} \; t(\tilde{\theta}) - \theta q(\tilde{\theta}).$$

This type of agency problem arises in many settings:

- interaction between the shareholders of a firm and its managers, or the firm and its workers: private information about the productivity of the managers or the workers;

- interaction between an investor and a firm, or a bank and its managers: private information about the projects undertaken;

- relationship between an insurance company and its customers: private information about the risks that the customer is facing;

- price discrimination: private information about the customers' willingness to pay.

The term "adverse selection" comes from the insurance market: when insurance companies raise their premiums, the first customers to drop out are those with the lowest risk; insurance companies are thus left with the highest-risk policyholders.[1]

[1] Interestingly, thanks to extensive databases, insurance companies now often have better information than their customers about the risks that they are facing – but this just turns the adverse-selection problem around.

1.2 A Simple Example

1.2.1 Price Discrimination

There are two parties:

- the principal is a firm (seller), which can produce a quantity q of a good at cost $C(q)$, such that $C', C'' > 0$;
- the agent is a customer (buyer), who obtains a utility θq from the good, where θ can take one of two values, $\underline{\theta}$ (with probability $\underline{\mu}$) or $\bar{\theta} > \underline{\theta}$ (with probability $\bar{\mu} = 1 - \underline{\mu}$).

Remark 1 *1. The variable q can also be interpreted as the "quality" of the good in question; the parameter θ can thus reflect the customer's size, or the taste for quality.*

2. The framework applies equally well to the case of a single customer (the probabilities $\underline{\mu}$ and $\bar{\mu}$ then reflecting the seller's prior beliefs about θ) and to the case of a large population of infinitesimal customers (in which case $\underline{\mu}$ and $\bar{\mu}$ can be interpreted as the proportion of customers with low and high values of θ).

1.2.2 Complete Information

Consider first a complete-information setting in which the seller knows the buyer's type. Denoting by t the total price paid by the buyer, the seller's profit-maximization problem can be written as:

$$\max_{t,q} t - C(q)$$
$$\text{s.t.} \quad \theta q - t \geq 0.$$

The constraint in this problem simply requires that the buyer should have a non-negative utility (since the buyer can always choose to buy nothing at all).

The seller maximizes her profit by setting t as high as possible. Thus the constraint is binding, $t = \theta q$, and we can rewrite the seller's problem as

$$\max_q \ \theta q - C(q).$$

This leads to (where the superscript FB stands for "First-Best"):

- the optimal quantity, $q^{FB}(\theta)$, solves the first-order condition $C'(q) = \theta$.
- the optimal transfer is then $t^{FB}(\theta) = \theta q^{FB}(\theta)$.

1.2.3 Incomplete Information

Let us now turn to a more realistic incomplete-information setting, where the seller no longer knows the buyer's true type, but only the relative probabilities $\underline{\mu}$ and $\bar{\mu}$. The seller will thus seek to maximize her *expected* profit, taking into account the participation or "Individual Rationality" constraint ("IR" hereafter) as well as the "Incentive Compatibility" constraints ("IC" hereafter); that is, the seller's profit-maximization problem becomes (where the overline and underline respectively refer to "high-type" and "low-type"):

$$\begin{aligned} \max_{(\underline{q},\underline{t}),(\bar{q},\bar{t})} \quad & \underline{\mu}(\underline{t} - c(\underline{q})) + \bar{\mu}(\bar{t} - c(\bar{q})) \\ \text{s.t.} \quad & \bar{\theta}\bar{q} - \bar{t} \geq 0 && (\overline{IR}) \\ & \underline{\theta}\underline{q} - \underline{t} \geq 0 && (\underline{IR}) \\ & \bar{\theta}\bar{q} - \bar{t} \geq \bar{\theta}\underline{q} - \underline{t} && (\overline{IC}) \\ & \underline{\theta}\underline{q} - \underline{t} \geq \underline{\theta}\bar{q} - \bar{t}. && (\underline{IC}) \end{aligned}$$

Denoting the buyer's rents (i.e., net utility) as $\bar{r} = \bar{\theta}\bar{q} - \bar{t}$ and $\underline{r} = \underline{\theta}\underline{q} - \underline{t}$, we can rewrite this problem as:

$$\begin{aligned} \max_{(\underline{q},\underline{r}),(\bar{q},\bar{r})} \quad & \underline{\mu}(\underline{\theta}\underline{q} - c(\underline{q}) - \underline{r}) + \bar{\mu}(\bar{\theta}\bar{q} - c(\bar{q}) - \bar{r}) \\ \text{s.t.} \quad & \bar{r} \geq 0 && (\overline{IR}) \\ & \underline{r} \geq 0 && (\underline{IR}) \\ & \bar{r} \geq \underline{r} + (\bar{\theta} - \underline{\theta})\underline{q} && (\overline{IC}) \\ & \underline{r} \geq \bar{r} - (\bar{\theta} - \underline{\theta})\bar{q}. && (\underline{IC}) \end{aligned}$$

Combining the two IC constraints yields:

$$(\bar{\theta} - \underline{\theta})\bar{q} \geq \bar{r} - \underline{r} \geq (\bar{\theta} - \underline{\theta})\underline{q},$$

which in turn implies

$$\bar{q} \geq \underline{q}. \tag{1.1}$$

That is, incentive compatibility implies that a customer with a higher type θ must obtain a higher q.

This maximization problem has four choice variables and four constraints. How do we solve this? Take the economist's approach: guess which constraints are binding, turn them into equalities, and omit the non-binding ones:

- Together, the IR for a low-type $(\overline{IR})$ and the IC for a high-type $(\underline{IC})$ imply the IR for high-type $\left(\overline{IR}\right)$; indeed, adding $\left(\overline{IR}\right)$ and $(\underline{IC})$ yields

$$\bar{r} \geq (\bar{\theta} - \underline{\theta})\underline{q} \geq 0,$$

so we can ignore $\left(\overline{IR}\right)$; intuitively, since a high-type who mimics a low-type obtains in this way more utility than a low-type, a high-type is always willing to participate if a low-type is.

- Ignoring $\left(\overline{IR}\right)$ implies in turn that:

  - $\left(\overline{IC}\right)$ must be binding: otherwise, the seller could increase her profit by slightly reducing $\bar{r}$ without violating any constraint;
  - using (1.1), this in turn implies that $(\underline{IC})$ can be ignored, since then $\bar{r} - \underline{r} = (\bar{\theta} - \underline{\theta})\underline{q} \leq (\bar{\theta} - \underline{\theta})\bar{q}$:
  - $(\underline{IR})$ must also be binding: otherwise, the seller could increase her profit by slightly reducing the rents of both types by the same amount, as this does not affect the IC constraints (since we are subtracting the same amount from both sides of the inequalities);

Thus, at the optimum, we can ignore $\left(\overline{IR}\right)$, replace both $\left(\overline{IC}\right)$ and $(\underline{IR})$ with equality constraints, and replace $(\underline{IC})$ with (1.1):

$$\begin{aligned} \max_{\underline{q},\bar{q}} \; & \underline{\mu}(\underline{\theta}\underline{q} - C(\underline{q}) - \underline{r}) + \bar{\mu}(\bar{\theta}\bar{q} - C(\bar{q}) - \bar{r}) \\ \text{s.t.} \quad & \underline{r} = 0, && (\underline{IR}) \\ & \bar{r} = (\bar{\theta} - \underline{\theta})\underline{q}, && (\overline{IC}) \\ & \bar{q} \geq \underline{q}. \end{aligned}$$

The two binding constraints determine the buyer's rents, as a function of the quantity assigned to a low type: $\underline{r} = 0$ and $\bar{r} = (\bar{\theta} - \underline{\theta})\underline{q}$; plugging these into the objective function leads to:

$$\begin{aligned} \max_{\underline{q},\bar{q}} \; & \underline{\mu}\left[\underline{\theta}\underline{q} - C(\underline{q})\right] + \bar{\mu}\left[\bar{\theta}\bar{q} - C(\bar{q}) - (\bar{\theta} - \underline{\theta})\underline{q}\right] \\ \text{s.t.} \quad & \bar{q} \geq \underline{q}. \end{aligned}$$

If we ignore for the moment the monotonicity constraint, the first-order conditions yield:

$$\begin{aligned} \text{w.r.t. } \bar{q}: \quad & C'(q) = \bar{\theta} \\ \text{w.r.t. } \underline{q}: \quad & C'(q) = \underline{\theta} - \frac{\bar{\mu}}{\underline{\mu}}(\bar{\theta} - \underline{\theta}). \end{aligned} \tag{1.2}$$

The first-order condition with respect to $\bar{q}$ is the same as for the first-best (with full information); hence the solution is $\bar{q}^{SB} = q^{FB}(\bar{\theta})$. By contrast, the first-order condition with respect to $\underline{q}$ implies that $\underline{q}^{SB} < q^{FB}(\underline{\theta})$. Since the quantity assigned to a low-type customer, $\underline{q}$, determines the rent $\bar{r}$ that needs to be left to a high-type agent, it is optimal for the seller to lower this quantity below the efficient level, so as to extract more surplus from a high-type.

Since $q^{FB}(\underline{\theta}) < q^{FB}(\bar{\theta})$, it follows that the candidate solutions characterized by the above first-order conditions satisfy $\bar{q}^{SB} > \underline{q}^{SB}$. These candidate solutions thus indeed constitute the second-best optimum. Computing the optimal rents in this second-best setting, we find

$$\begin{aligned} \underline{r}^{SB} &= \underline{r}^{FB} = 0, \\ \bar{r}^{SB} &= (\bar{\theta} - \underline{\theta})\underline{q}^{SB} > 0. \end{aligned}$$

Using $t(\theta) = \theta q(\theta) - r(\theta)$, the transfers are then:

$$\begin{aligned} \underline{t}^{SB} &= \underline{\theta}\underline{q}^{SB}, \\ \bar{r}^{SB} &= \bar{\theta}\bar{q}^{FB} - (\bar{\theta} - \underline{\theta})\underline{q}^{SB} = \underline{\theta}\underline{q}^{SB} + \bar{\theta}\left(\bar{q}^{FB} - \underline{q}^{SB}\right). \end{aligned}$$

Remark: corner solution. We implicitly assumed that the seller finds it optimal to sell to both types of buyers. However, if the quantity $C'(0) < \underline{\theta} - \frac{\bar{\mu}}{\underline{\mu}}(\bar{\theta} - \underline{\theta})$ (e.g., if $\underline{\theta} - \frac{\bar{\mu}}{\underline{\mu}}(\bar{\theta} - \underline{\theta})$ is negative), then it is optimal for the seller to ignore low-type buyers; in that case, the seller can extract all the surplus from high-type buyers (since $\underline{q} = 0$ leads to $\bar{r} = 0$), and thus will maintain the efficient level of trade ($\bar{q}^{SB} = \bar{q}^{FB}$).

Remark: rent vs. efficiency trade-off. It *is* possible for the seller to implement the first-best, since it satisfies the monotonicity requirement: $q^{FB}(\bar{\theta}) \geq q^{FB}(\underline{\theta})$. However, it is not optimal for the seller to do so, because of the expected rent she would have to pay is greater: while the rent to a low-type buyer is zero ($\underline{r} = 0$), the rent to a high-type buyer is determined by the quantity assigned to a low type, and increases with that quantity; thus, implementing the first-best would require giving a high-type buyer a rent equal to:

$$\bar{r}^{FB} = (\bar{\theta} - \underline{\theta})q^{FB}(\underline{\theta}) > (\bar{\theta} - \underline{\theta})\underline{q}^{SB} = \bar{r}^{SB}.$$

Furthermore, keeping in mind that the seller's objective can be expressed as total expected surplus minus expected rent, consider the impact of a small reduction of the quantity assigned to a low-type, just below the first-best level $q^{FB}(\underline{\theta})$: this generates only a second-order loss of the surplus associated with a low type, but triggers a first-order reduction of the rent that must be left to a high type; it follows that, while the first-best is implementable, it is optimal for the seller to depart from it and reduce the quantity assigned to a low-type buyer below $q^{FB}(\underline{\theta})$, so as to save on the rent left to a high-type buyer.

Remark: implementation. The above optimal mechanism can be implemented as a non-linear tariff, with a marginal price equal to $\underline{\theta}$ up to $\underline{q}^{SB}$, and jumping to $\bar{\theta} > \underline{\theta}$ afterwards (see Figure 13.1); indeed, when confronted with this non-linear tariff:

- a low-type customer is willing to choose $\underline{q}^{SB}$ (he is actually indifferent between all levels $q \leq \underline{q}^{SB}$ – a slight reduction in the marginal price would suffice to break this indifference and induce $\underline{q}^{SB}$ for sure);
- a high-type customer is willing to buy $\bar{q}^{SB}$ (he is indifferent between all levels $q > \underline{q}^{SB}$; to break this indifference, the marginal price should be slightly reduced for $q \leq \bar{q}^{FB}$ and slightly increased afterwards).

Note that this tariff is convex; thus, it cannot be implemented by a family of two-part tariffs (as the lower envelope of a family of two-part tariffs is necessarily concave). It could however be implemented using three-part tariffs.[2] See Figure 13.3.

Remark: commitment. The above analysis relies on the seller's ability to commit to a given mechanism. It may no longer be credible if the seller can "renegotiate", unilaterally or bilaterally, the contracting terms.

. *Unilateral renegotiation.* The above mechanism commits the seller to leave an informational rent to a high-type buyer. Ex ante, this rent is needed to induce the buyer to reveal his type. But once the type has been revealed, the seller has an incentive to renege on her promise and charge the "full" price to the buyer; that is, if the buyer chooses the option $(\bar{q}, \bar{t})$, then the seller knows that the buyer has a high type $\bar{\theta}$, and she can charge him $\tilde{t} = \bar{\theta}\bar{q}$. Of course, if such opportunistic behaviour is anticipated, then a high-type buyer will not reveal his type and will instead pretend having a low type; in other words, if the seller cannot credibly commit herself to "pay" for the information provided by the buyer, then the buyer will not communicate this information.

[2] Cell phone plans are common examples of three-part tariffs, as they often (1) require the subscriber to pay a fixed monthly fee, (2) provide a certain amount of minutes at no additional expense, and (3) charge some linear price for minutes beyond this amount. See Figure 13.2.

. *Bilateral renegotiation.* Another problem appears if the two parties cannot commit not to renegotiate (bilaterally and voluntarily) the terms of the mechanism. From an ex ante perspective, it is optimal to commit to an inefficiently low level of trade with a low-type buyer ($\underline{q}^{SB} < q^{FB}(\underline{\theta})$), in order to reduce the rent that must be left to a high-type buyer. But then, if the buyer selects the option designed for a low type, it becomes common knowledge that the buyer has a low type, and it is then in the best interest of the buyer *and* the seller to renegotiate the terms of the contract and replace the second-best level of trade, $\underline{q}^{SB}$, with the efficient (i.e. first-best) one, $q^{FB}(\underline{\theta})$. This increases the total surplus by $\Delta W = W\left(q^{FB}(\underline{\theta})\right) - W\left(q^{SB}(\underline{\theta})\right) > 0$, where $W = \theta q - c(q)$, and the two parties can split this additional surplus as they wish by adjusting the transfer to $t = \underline{t}^{SB} + \alpha \Delta W$, where $\alpha \in [0, 1]$ denotes the share of the gain ΔW that accrues to the seller. Of course, if this renegotiation is anticipated, then a high-type buyer will anticipate that reporting a low type will lead to $q^{FB}(\underline{\theta}) > \underline{q}^{SB}$ rather than to $\underline{q}^{SB}$, and thus a larger rent (based on $q^{FB}(\underline{\theta})$ rather than on $\underline{q}^{SB}$) will have to be paid to keep the buyer reporting a high type. Thus, if the seller is unable to commit not to renegotiate, even bilaterally, then the information becomes more costly to acquire; this, in turn, may imply that full information disclosure is no longer optimal. That is, while under full commitment the revelation principle tells us that, without loss of generality, one can restrict attention to mechanisms where the buyer truthfully reports his type, under limited commitment, it may become optimal to elicit less information.

1.3 A More General Treatment

1.3.1 Framework

The agent has a utility $u(q, t; \theta)$, where $q \in Q$ represents a "real" dimension (e.g., the volume of trade, the quality of a project, the amount of public good being provided, etc.), $t \in \mathbb{R}$ is the transfer to the principal, and θ is the agent's type; this utility is quasilinear with respect to the transfer t:

$$U(q, t; \theta) = V(q; \theta) - t.$$

The agent's type is distributed over $\Theta = [\underline{\theta}, \bar{\theta}]$ according to a density $f(\cdot)$ and cdf $F(\cdot)$. The principal's utility is given by $t - C(q, \theta)$.

Under complete information, the principal solves

$$\max_{q,t} t - C(q, \theta)$$
$$\text{s.t } V(q; \theta) - t \geq 0.$$

Since the principal wishes to maximize t, the agent's participation constraint will be binding; thus, for a given q, the principal sets the transfer to $t = V(q;\theta)$ and thus chooses q so as to maximize total surplus:

$$\max_q W(q;\theta) = V(q;\theta) - C(q;\theta).$$

Assuming that W is concave in q, the first-best level of q, $q^{FB}(\theta)$, is such that:

$$\frac{\partial V}{\partial q}(q;\theta) = \frac{\partial C}{\partial q}(q;\theta).$$

The corresponding transfer is then $t^{FB}(\theta) = V(q^{FB}(\theta), \theta)$.

From now on, we will consider an incomplete information setting in which only the agent knows his type; the principal must therefore account for incentive compatibility. To characterize the optimal contract, we will decompose the problem into two steps: first, we will characterize the set of "feasible" (i.e., incentive-compatible and individual rational) allocations – this is the "implementation" stage; second, we will look for the optimal allocation within this feasibility set – the "optimization" stage.

1.3.2 Implementation

An allocation rule f can be written as

$$\begin{array}{rcl} f: \ \Theta & \rightarrow & A = Q \times \mathbb{R} \\ \theta & \mapsto & f(\theta) = (q(\theta), t(\theta)) \end{array}$$

Revelation Principle

From the Revelation Principle, we know that any implementable allocation must be incentive compatible; that is, if there exists a revelation mechanism $(M, g: M \rightarrow A)$ that triggers a response $h: \Theta \rightarrow M$ implementing f (that is, such that $f = g \circ h$), then it must satisfy: $\forall \theta, \tilde{\theta} \in \Theta$, $U(f(\theta), \theta) \geq U\left(f\left(\tilde{\theta}\right), \theta\right)$. Conversely, since there is a single agent here (so that the concepts of dominant strategy, Bayesian, and Nash equilibrium coincide), the multiplicity of responses is not too troublesome (since it is not too much of an issue for the implementation in dominant strategies). Therefore, in what follows we will restrict our attention to direct mechanisms that are incentive-compatible.

Letting

$$r(\theta) = V(q(\theta);\theta) - t(\theta)$$

denote the agent's *rent*, the relevant constraints can be expressed as:

$$r(\theta) \geq 0,$$
$$r(\theta) = \max_{\tilde{\theta}} V(q(\tilde{\theta});\theta) - t(\tilde{\theta}).$$

We will make the following assumption, known as the **Spence-Mirrlees condition** (a.k.a. the **single-crossing property**):

Assumption: $\forall \theta \in \Theta, \forall q \in Q,$

$$\partial^2_{\theta q} V(q;\theta) \geq 0. \qquad ((SM_+))$$

That is, the higher the agent's type θ, the higher his marginal utility; that is, an increase in the agent's type means that the agent is willing to trade more, and this is true at all levels of trade q. This assumption translates into requiring a constant sign on the second partial derivative $\partial^2_{\theta q} V$, i.e., requiring the marginal utility to be monotone in θ. In the Spence-Mirrlees condition stated above, the marginal utility is monotonically increasing; however, the single-crossing property is equally valid for a monotonically decreasing marginal utility: we will refer to the former as the Spence-Mirrlees condition with positive sign (SM_+), and the latter as the Spence-Mirrlees condition with a negative sign: $\forall \theta \in \Theta, \forall q \in Q$, $\partial^2_{\theta q} V(q;\theta) \leq 0$ $(SM\ )$. The following theorem provides a characterization of incentive-compatibility under the Spence Mirrlees condition:

Theorem 1 *Assume the Spence-Mirrlees condition hods with a positive sign (SM_+). Then $(q(\cdot), r(\cdot))$ is incentive-compatible if and only if*

$$\begin{cases} r(\theta) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \partial_\theta V(q(s);s)\, ds, \\ \quad q(\theta) \text{ is non-decreasing.} \end{cases}$$

Proof. We prove the two implications in sequence.

Step 1: $\Rightarrow$) Suppose that the Spence-Mirrlees condition holds with a positive sign, and assume that $(q(\cdot), r(\cdot))$ is incentive-compatible. Incentive compatibility for type θ implies

$$V(q(\theta);\theta) - t(\theta) \geq V(q(\tilde{\theta});\theta) - t(\tilde{\theta}),$$

which we can write in terms of rent as

$$r(\theta) \geq r(\tilde{\theta}) + V(q(\tilde{\theta});\theta) - V(q(\tilde{\theta});\tilde{\theta}). \qquad (1.3)$$

Similarly, incentive compatibility for type $\tilde{\theta}$ implies

$$V(q(\tilde{\theta});\tilde{\theta}) - t(\tilde{\theta}) \geq V(q(\theta);\tilde{\theta}) - t(\theta),$$

which in terms of rent yields

$$r(\tilde{\theta}) \geq r(\theta) + V(q(\theta); \tilde{\theta}) - V(q(\theta); \theta). \tag{1.4}$$

Combining (1.3) and (1.4) leads to

$$V(q(\tilde{\theta}); \tilde{\theta}) - V(q(\tilde{\theta}); \theta) \geq r(\tilde{\theta}) - r(\theta) \geq V(q(\theta); \tilde{\theta}) - V(q(\theta); \theta),$$

and thus (rewriting the two outer expressions using integrals):

$$\begin{aligned}
\int_{\theta}^{\tilde{\theta}} \partial_\theta V(q(\tilde{\theta}); s)\, ds \quad &\geq \quad \int_{\theta}^{\tilde{\theta}} \partial_\theta V(q(\theta); s)\, ds \\
&\Longleftrightarrow \quad \int_{\theta}^{\tilde{\theta}} \left[\partial_\theta V(q(\tilde{\theta}); s) - \partial_\theta V(q(\theta); s)\right] ds \geq 0 \\
&\Longleftrightarrow \quad \int_{\theta}^{\tilde{\theta}} \int_{q(\theta)}^{q(\tilde{\theta})} \partial^2_{\theta q} V(q, s)\, dq\, ds \geq 0.
\end{aligned}$$

Since the integrand $\partial^2_{\theta q} V(q, s)$ is positive from the Spence-Mirrlees condition (SM_+), and the value of the integral is non-negative, it follows that the boundaries of the two integrals must move in the same direction (that is, $\tilde{\theta} > \theta$ implies $q(\tilde{\theta}) \geq q(\theta)$); therefore, $q(\cdot)$ is non-decreasing.

This in turn implies that $q(\cdot)$ is continuous almost everywhere; it follows that

$$r(\theta) = \max_{\tilde{\theta}} V(q(\tilde{\theta}); \theta) - t(\tilde{\theta})$$

is almost continuously continuously differentiable, and its derivative satisfies

$$\dot{r}(\theta) = \partial_\theta V(q(\theta); \theta).$$

Integrating both sides of this equation from $\underline{\theta}$ to θ, we obtain

$$r(\theta) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \partial_\theta V(q(s); s)\, ds,$$

which completes the proof of step 1.

Step 2: $\Leftarrow$) Define $\varphi(\tilde{\theta}; \theta)$ to be the payoff that an agent of type θ gets from claiming to be type $\tilde{\theta}$:

$$\varphi(\tilde{\theta}; \theta) \equiv V(q(\tilde{\theta}); \theta) - t(\tilde{\theta}).$$

We have:

$$\begin{aligned}\varphi(\theta;\theta)-\varphi(\tilde{\theta};\theta) &= [V(q(\theta);\theta)-t(\theta)]-\left[V(q(\tilde{\theta});\theta)-t(\tilde{\theta})\right]\\ &= r(\theta)-\left[r(\tilde{\theta})+V(q(\tilde{\theta});\theta)-V(q(\tilde{\theta});\tilde{\theta})\right]\\ &= \left[r(\theta)-r(\tilde{\theta})\right]-\left[V(q(\tilde{\theta});\theta)-V(q(\tilde{\theta});\tilde{\theta})\right]\\ &= \int_{\tilde{\theta}}^{\theta}\partial_\theta V(q(s);s)\,ds-\int_{\tilde{\theta}}^{\theta}\partial_\theta V(q(\tilde{\theta});s)\,ds,\end{aligned}$$

where the last equality stems in part from the assumption ($r(\theta)=r(\underline{\theta})+\int_{\underline{\theta}}^{\theta}\partial_\theta V(q(s);s)\,ds$) and in part from the identity $V(q(\tilde{\theta});\theta)-V(q(\tilde{\theta});\tilde{\theta})=\int_{\tilde{\theta}}^{\theta}\partial_\theta V(q(\tilde{\theta});s)\,ds$. Rewriting the last expression as a double integral leads to

$$\begin{aligned}\varphi(\theta;\theta)-\varphi(\tilde{\theta};\theta) &= \int_{\tilde{\theta}}^{\theta}\left[\partial_\theta V(q(s);s)\,ds-\partial_\theta V(q(\tilde{\theta});s)\right]\,ds\\ &= \int_{\tilde{\theta}}^{\theta}\int_{q(\tilde{\theta})}^{q(s)}\partial^2_{\theta q}V(q;s)\,dq\,ds\\ &\geq 0,\end{aligned}$$

where the inequality follows from (SM_+). Thus, an agent of type θ maximizes his net payoff by truthfully revealing his type. ■

Note the similarity between this analysis and our analysis in the example of price discrimination above:

- In both instances, we use the incentive-compatibility conditions to relate the evolution of the agent's rent, $r(\theta)$, to the quantity profile $q(\theta)$; more specifically, thanks to the Spence-Mirrlees condition, incentive-compatibility boils down to a monotonicity requirement ($q(\cdot)$ must be non-decreasing), and to $\dot{r}(\theta)=\partial_\theta V(q(\theta);\theta)$.

- In the previous two-type setting $\left(\Theta=\{\underline{\theta},\bar{\theta}\}\right)$ where $V(q;\theta)=\theta q$, we had similarly $\bar{q}\geq\underline{q}$ and

$$\bar{r}-\underline{r}=\Delta\theta\times\underline{q}=\int_{\underline{\theta}}^{\bar{\theta}}\partial_\theta(\theta q(s))\,ds.$$

- Likewise, with more than 2 types, $\theta_1<\theta_2<...<\theta_n$, the sequence of outputs should be (weakly) increasing ($q_1\leq q_2\leq\ldots\leq q_n$) and the rents should satisfy $r_{k+1}-r_k=(\theta_{k+1}-\theta_k)q_k$.

1.3.3 Optimization

Having determined the set of feasible mechanisms, we now seek to characterize the optimal one.

Under complete information, the first-best outcome is $(q^{FB}(\theta), t^{FB}(\theta))$, where $q^{FB}(\theta)$ solves

$$\partial_q V(q;\theta) = \partial_q C(q;\theta),$$

and where $t^{FB}(\theta) = V(q^{FB}(\theta), \theta)$; that is, under the first-best, the principal extracts all the agent's surplus ($r^{FB}(\theta) = 0$), and thus chooses q so as to maximize the total surplus $W(q;\theta)$.

Under incomplete information, the principal solves

$$\begin{aligned}
&\max_{q(\cdot),t(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [t(\theta) - c(q(\theta);\theta)]\, f(\theta)\, d\theta \\
&\text{s.t.} \quad V(q(\theta);\theta) - t(\theta) \geq 0 \qquad \forall \theta \\
&\qquad\quad V(q(\theta);\theta) - t(\theta) \geq V(q(\tilde{\theta});\theta) - t(\tilde{\theta}) \qquad \forall \theta, \tilde{\theta}.
\end{aligned}$$

Using the rent $r(\theta) = V(q(\theta);\theta) - t(\theta)$ and the above analysis, we can rewrite this maximization problem as

$$\max_{q(\cdot),r(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} [V(q(\theta);\theta) - c(q(\theta);\theta) - r(\theta)]\, f(\theta)\, d\theta$$

$$\text{s.t.} \quad \forall \theta \in \Theta, \quad r(\theta) \geq 0, \tag{1.5}$$

$$\forall \theta \in \Theta, \quad r(\theta) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \partial_\theta V(q(s);s)\, ds, \tag{1.6}$$

$$q(\cdot) \text{ is increasing.} \tag{1.7}$$

Let us introduce one additional monotonicity assumption:

Assumption: $\forall \theta \in \Theta, \forall q \in Q$,

$$\partial_\theta V(q;\theta) \geq 0. \tag{(M_+)}$$

Together with (SM_+), these two assumptions assert an increase in the type θ has a similar qualitative impact on the agent's utility (and thus on his willingness to participate) and his marginal utility (and thus on how much he is willing to trade).

Since the principal wants to minimize the agent's rents, at least one participation constraint must be binding (otherwise, the principal could decrease all rents uniformly without affecting neither the incentive constraints nor the monotonicity requirement). Conversely, under the assumption $\partial_\theta V \geq 0$, every type of agent is willing to participate whenever the lowest type is willing to do so. It follows that the (only) binding participation constraint is that of the lowest type $\underline{\theta}$: $r(\underline{\theta}) = 0$. The incentive constraints then boil down to $(q(\cdot) \nearrow$ and)

$$r(\theta) = \int_{\underline{\theta}}^{\theta} \partial_\theta V(q(s); s)\, ds.$$

Plugging this back into the principal's objective function, we can rewrite her problem as:

$$\max_{q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V(q(\theta);\theta) - c(q(\theta);\theta) - \int_{\underline{\theta}}^{\theta} \partial_\theta V(q(s); s)\, ds \right] f(\theta)\, d\theta \tag{1.8}$$

$$\text{s.t.} \quad q(\theta) \text{ is increasing.}$$

We can use Fubini's theorem to switch the order of integration in the double integral (see Figure 1.1); the integral then simplifies to

$$\begin{aligned}\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} \partial_\theta V(q(s); s)\, ds\, f(\theta) d\theta &= \int_{\underline{\theta}}^{\bar{\theta}} \left[\int_{s}^{\bar{\theta}} f(\theta)\, d\theta \right] \partial_\theta V(q(s); s)\, ds \\ &= \int_{\underline{\theta}}^{\bar{\theta}} [1 - F(s)]\, \partial_\theta V(q(s); s)\, ds \\ &= \int_{\underline{\theta}}^{\bar{\theta}} [1 - F(\theta)]\, \partial_\theta V(q(\theta); \theta)\, d\theta,\end{aligned}$$

where the final step simply reflects a cosmetic change for the notation of the generic variable of integration from s to θ.

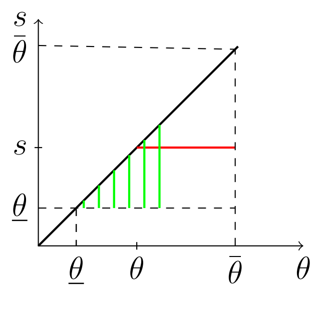

Figure 1.1: Fubini's theorem

Plugging this result back into the principal's objective expressed in (1.8) yields

$$\int_{\underline{\theta}}^{\bar{\theta}} [V(q(\theta);\theta) - c(q(\theta);\theta)]\, f(\theta)\, d\theta - \int_{\underline{\theta}}^{\bar{\theta}} (1 - F(\theta))\partial_\theta V(q(\theta);\theta)\, d\theta$$

$$= \int_{\underline{\theta}}^{\bar{\theta}} \{[V(q(\theta);\theta) - c(q(\theta);\theta)]\, f(\theta) - (1 - F(\theta))\partial_\theta V(q(\theta);\theta)\}\ d\theta$$

$$= \int_{\underline{\theta}}^{\bar{\theta}} [V(q(\theta);\theta) - c(q(\theta);\theta) - \gamma(\theta)\partial_\theta V(q(\theta);\theta)]\, f(\theta)\, d\theta, \qquad (1.9)$$

where $\gamma(\theta) \equiv \frac{1-F(\theta)}{f(\theta)}$ denote the hazard rate. Ignoring the monotonicity requirement ($q(\cdot) \nearrow$) yields a candidate solution, $\hat{q}(\theta)$, which, for any given θ, simply maximizes the integrand and is thus characterized by the first-order condition

$$\partial_q V(\hat{q};\theta) - \partial_q C(\hat{q};\theta) = \gamma(\theta)\partial^2_{\theta q} V(\hat{q};\theta).$$

By construction, if the solution to an unconstrained program satisfies the omitted constraint – namely here, the condition $q(\cdot) \nearrow$ –, then it is also the solution to the constrained program. Therefore, if $\hat{q}(\cdot)$ is increasing, then $q^{SB}(\theta) = \hat{q}(\theta)$.

If instead $\hat{q}(\cdot)$ is *not* increasing, then there must be some bunching: the second-best must be constant for at least a subset of types. Guesnerie-Laffont(*JPE* 1986) provide the following characterization:

- $q^{SB}(\theta) = \hat{q}(\theta)$ whenever $\dot{q}^{SB}(\theta) > 0$; that is, whenever it increases, the second-best solution q^{SB} coincides with $\hat{q}$.

- $\bar{q} = \arg\max_q \int_{\theta_1}^{\theta_2} [V(q;\theta) - c(q;\theta) - \gamma(\theta)\partial_\theta V(q;\theta)]\, f(\theta)\, d\theta$; that is, in a bunching interval (where the second-best is thus constant), the output is set so as to maximize the principal's virtual objective given by (1.9) over that interval.

- q^{SB} is continuous; this provides additional conditions leading to the determination of the boundaries of the bunching intervals.

See Figures 1.2 and 1.3.

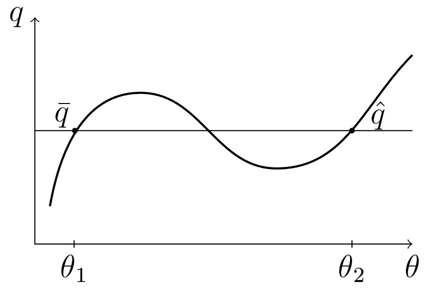

Figure 1.2: Monotonicity of q^{SB} and bunching

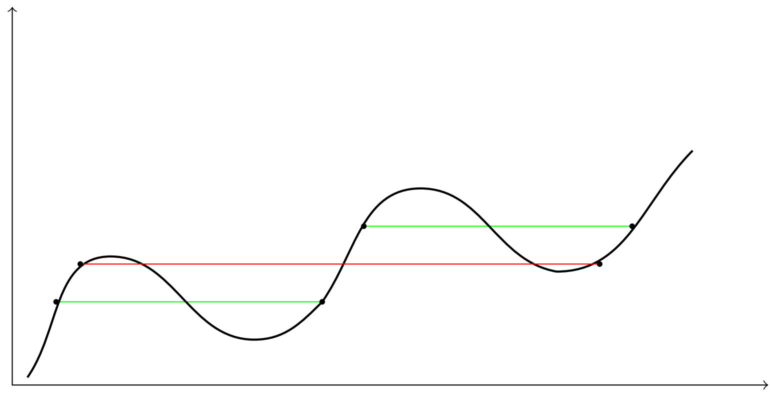

Figure 1.3: Boundaries of bunching

1.3.4 Examples

Price Discrimination

Consider an interaction between a firm and a consumer. The consumer's utility is given by

$$U(q,t;\theta) = \theta V(q) - t,$$

where $V'(q) > 0 > V''(q)$ and $V'(0) = +\infty$, whereas the cost function of the firm is $C(q) = cq$.

Using (1.9), the maximization problem of the firm can be written as:

$$\begin{array}{ll} \max\limits_{q(\cdot)} & \displaystyle\int_{\underline{\theta}}^{\bar{\theta}} \left[\theta V(q(\theta)) - c(q(\theta)) - \gamma(\theta) V(q(\theta))\right] f(\theta)\, d\theta, \\ \text{s.t.} & q(\cdot) \text{ is increasing} \end{array}$$

where $\gamma(\theta) = \frac{1-F(\theta)}{f(\theta)}$ denotes the likelihood ratio. Ignoring the monotonicity requirement, the first-order condition with respect to $q(\theta)$ yields

$$(\theta - \gamma(\theta))V'(\hat{q}) - c = 0,$$

or

$$\hat{q}(\theta) = V'^{-1}\left(\frac{c}{\theta - \gamma(\theta)}\right).$$

Suppose that the following assumption holds:

Assumption (Monotone Likelihood Ratio Property – $MLRP$**):** $\gamma(\theta) = \frac{1-F(\theta)}{f(\theta)}$ decreases as θ increases.

Since $V'^{-1}(\cdot)$ is decreasing, $MLRP$ guarantees that $\hat{q}(\theta)$ is increasing in θ and thus satisfies the monotonicity requirement; it follows that $q^{SB}(\theta) = \hat{q}(\theta)$.

The associated transfer is given by

$$\begin{aligned} t^{SB}(\theta) &= \theta V(q^{SB}(\theta)) - r^{SB}(\theta) \\ &= \theta V\left(q^{SB}(\theta)\right) - \int_{\underline{\theta}}^{\theta} V(q(s))\,ds. \end{aligned}$$

Note that this implies:

$$\dot{t}^{SB}(\theta) = \theta V'\left(q^{SB}(\theta)\right)\dot{q}^{SB}(\theta).$$

To implement this second-best, the firm can offer a family of options of the form $\{q(\theta), t(\theta)\}_{\theta\in\Theta}$. Alternatively, this can take the form of a single, non-linear tariff $t = \hat{T}(q)$ satisfying

$$t^{SB}(\theta) = \hat{T}\left(q^{SB}(\theta)\right).$$

Differentiating this equation with respect to θ yields $\dot{t}(\theta) = \hat{T}'(q(\theta))\dot{q}(\theta)$, and thus:

$$\hat{T}'(q) = \frac{\dot{t}(\theta)}{\dot{q}(\theta)} = \theta V'\left(q^{SB}(\theta)\right) = \frac{\theta c}{\theta - \gamma(\theta)} = \frac{c}{1 - \gamma(\theta)/\theta},$$

which is decreasing in θ under $MLRP$. Thus, the tariff $\hat{T}$ is concave, which in turn implies that it can be replicated as a family of affine (i.e., two-part) tariffs $\{\tau_\theta(q)\}_{\theta\in\Theta}$, of the form

$$\tau_\theta(q) = A(\theta) + p(\theta)q,$$

where $p(\theta) = \frac{c}{1-\gamma(\theta)/\theta}$ is decreasing in θ, and $A(\theta) = t^{SB}(\theta) - p(\theta)q^{SB}(\theta)$ is increasing in θ. See Figure 1.4.

Remark: market size. If $\underline{\theta} < \gamma(\underline{\theta})$ or $V'(0) < \frac{c}{\underline{\theta}-\gamma(\underline{\theta})}$, then some low types are excluded:

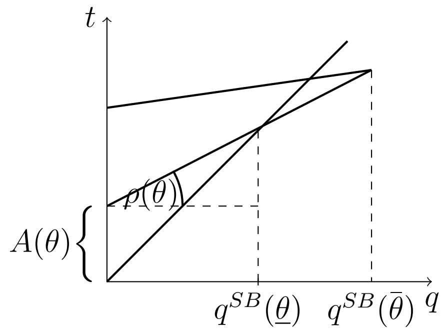

Figure 1.4: Two - part tariffs

- in the first-best, $q^{FB}(\theta) = 0$ for $\theta \leq \hat{\theta}^{FB}$: $\theta V'(0) = c$;
- in the second-best, $q^{SB}(\theta) = 0$ for $\theta \leq \hat{\theta}^{SB}$, where $\hat{\theta}^{SB} > \hat{\theta}^{FB}$ is characterized by: $(\theta - \gamma(\theta))\, V'(0) = c$; thus, $\hat{\theta}^{SB} - \gamma\left(\hat{\theta}^{SB}\right) = \hat{\theta}^{FB}\ (= c/V'(0))$, or $\hat{\theta}^{SB} = \hat{\theta}^{FB} + \gamma\left(\hat{\theta}^{SB}\right) > \hat{\theta}^{FB}$.

To illustrate this, consider the case $V(q) = q - q^2/2$; we then have:

- First-best: for $\theta \geq \hat{\theta}^{FB} = c$, $q^{FB}(\theta) = 1 - c/\theta$;
- First-best: for $\theta \geq \hat{\theta}^{SB} = c$, $q^{SB}(\theta) = 1 - c/(2\theta - 1)$;

As usual, there is no distortion at the top ($\gamma\left(\bar{\theta}\right) = 0$ implies $q^{SB}\left(\bar{\theta}\right) = q^{FB}\left(\bar{\theta}\right)$); as θ decreases below $\bar{\theta}$, the distortion increases, and the second-best level is equal to 0 for a wider range of types. See Figure 1.5.

Regulation

The following example is taken from Baron & Myerson (*Eca* 1982). The agent is a firm with payoff

$$U(q, t; \theta) = t - \theta q,$$

whereas the principal is a regulator with an objective function given by

$$W(q, t, \theta) = S(q) - (1 + \lambda)t + U = S(q) - \theta q - \lambda t.$$

- Under complete information (first-best, in which the regulator knows the firm's type θ), the optimal outcome solves $S'(q) = (1 + \lambda)\theta$.

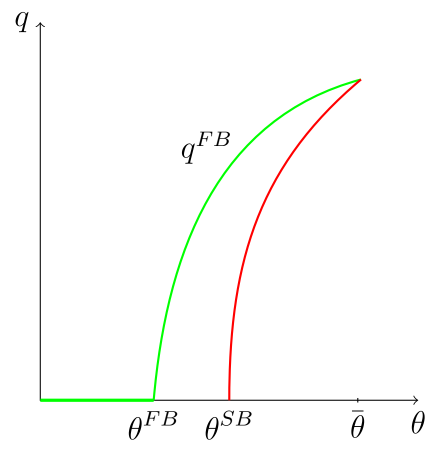

Figure 1.5: Distortion of the second best quantity

- Under incomplete information (second-best, in which the firm's type is the private information of the firm), $q^{SB}(\theta)$ solves $S'(q) = (1+\lambda)\theta + \lambda\frac{F(\theta)}{f(\theta)}$ (note that SM_- holds, and $\partial_\theta V < 0$: that is, the "good types" are the low ones here). See Figure 1.6.

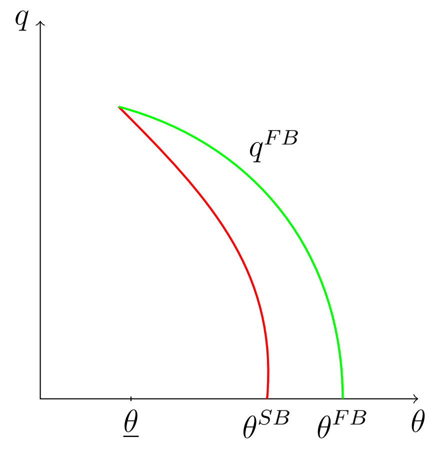

Figure 1.6: Distortion of the second best solution when the "good types" are the low ones

Labor-Managed Firm

Suppose that the output of the firm depends on its type θ and on the amount of labor l: $q = \theta f(l)$, with $f'' < 0 < f'$, and that the firm aims now at maximizing the added value per worker:

$$U(l, t; \theta) = \frac{\theta f(l) - K + t}{l},$$

where K denotes a fixed cost that needs to be recouped; the objective of the regulator is now

$$W(l, t, \theta) = \theta f(l) - K - wl,$$

where w denotes the opportunity cost of employing an additional worker in the firm.

In the first-best, the allocation of labor $l^{FB}(\theta)$ is increasing in θ, which is intuitive: it is optimal to assign more labor to more productive firms. But the Spence-Mirrlees condition is

$$\partial^2_{\theta l} U = \partial_l \left(\frac{f(l)}{l} \right) < 0.$$

It follows that only *decreasing* labor profiles can be implemented; as a result of this naked divergence of objectives, there is complete bunching ("non-responsiveness"): the second-best optimum consists in allocating a constant amount of labor, regardless of the productivity θ.

c L

1.4 Variations

1.4.1 Multiple Outputs

Let the output q be a vector $q = (q_1, \ldots, q_n)$. The objectives of the agent and of the principal respectively become $U(q, t; \theta) = V(q_1, \ldots, q_n; \theta) - t$ and $t - C(q_1, \ldots, q_n; \theta)$. What are the incentive-compatibility conditions here? Suppose the Spence-Mirrlees condition with positive sign holds for each output. Then, adapting the proof of Theorem 1, an output profile can be implemented (with adequate transfers) if $\partial^2_{\theta q_i} V(q(\theta); \theta) \geq 0$ and $q_i(\theta)$ is increasing for *each* $i = 1, ..., n$. These conditions are however sufficient, but not necessary. Following the steps of the proof of Theorem 1,we have:

$$\begin{aligned}\varphi(\theta,\theta)-\varphi(\tilde{\theta},\theta) &= r(\theta)-r(\tilde{\theta})-\left[V(q(\tilde{\theta});\theta)-V(q(\tilde{\theta});\tilde{\theta})\right]\\ &= \int_{\tilde{\theta}}^{\theta}\partial_\theta V(q(\theta);s)\,ds-\int_{\tilde{\theta}}^{\theta}\partial_\theta V(q(\tilde{\theta});s)\,ds\\ &= \int_{\tilde{\theta}}^{\theta}\int_{s}^{\theta}\frac{d}{d\hat{\theta}}\left(\partial_\theta V(q(\hat{\theta});s)\right)\,d\hat{\theta}\,ds.\end{aligned}$$

A sufficient condition to ensure that this expression is non-negative (so that $\varphi(\theta,\theta)-\varphi(\tilde{\theta},\theta)\geq 0$, which implies that incentive compatibility holds) is

$$\frac{d}{d\theta}(\partial_\theta V(q(\theta);s))\geq 0,$$

that is:

$$\sum_{i=1}^{n}\partial^2_{\theta q_i}V(q(\theta);s)\frac{dq_i}{d\theta}(\theta)\geq 0.$$

Assuming that the Spence-Mirrlees condition holds with a positive sign for each $i=1,...,n$, this inequality is satisfied if each of the derivatives $dq_i/d\theta$ is non-negative; however, the inequality can still be satisfied even if some of these derivatives are negative, so long as the total sum is non-negative.

Example

The following example comes from Laffont & Tirole (*JPE* 1986). Consider the following interaction between a firm (the principal) and an agent. The agent's cost is given by

$$c(e,\theta)q+\psi(e),$$

where q the quantity produced, θ the productivity of the firm (i.e., the agent's type), $c(e,\theta)=\theta-e$ represents the audited cost, which is observable and can thus be contracted upon, whereas e denotes the managerial effort and is not observed by outsiders. The regulator and the firm can thus contract on q, t, and c, and the payoff of the firm is given by

$$t-cq-\psi(e)=t-cq-\psi(\theta-c),$$

which thus fits the above "multiple output" framework.

1.4.2 Noisy Observations

Suppose now that the output q is not perfectly observed; that is, what is only observed is a noisy version of it:

$$\tilde{q} = q + \varepsilon,$$

where ε is a "white noise": $\mathbb{E}[\varepsilon] = 0$. If agents could contract directly on the true value q, then we have seen that the principal and the agent could achieve the second-best through some non-linear "tariff" $t = T^{SB}(q)$. What if only the noisy observation $\tilde{q}$ is observable?

(a) If the optimal tariff $T^{SB}(q)$ is concave, then it can be replicated with a family of two-part tariffs: $\left\{\tau_\theta^{SB}(q) = A(\theta) + p(\theta)q\right\}_{\theta\in\Theta}$. But when facing a two-part tariff $A(\theta) + p(\theta)q$, choosing a given output level q yields an expected transfer which exactly coincides with the transfer that would apply in the absence of any noise:

$$\mathbb{E}[\tau_\theta^{SB}(\tilde{q})] = \mathbb{E}[\tau_\theta^{SB}(q+\varepsilon)] = \mathbb{E}[A(\theta)+p(\theta)(q+\varepsilon)] = A(\theta)+p(\theta)q = \tau_\theta^{SB}(q).$$

Therefore, from the point of view of the agent, even though the value of q is observed with noise, the expected payment is exactly the one that would be made if q were perfectly observed and the contract were made directly on q. It follows that the family of two-part tariffs, $\left\{\tau_\theta^{SB}(q)\right\}_{\theta\in\Theta}$, remains optimal and as efficient as in the absence of noise; that is, the fact that the output is only observed with a noise does not affect the principal-agent relationship. [While we consider the case of a white noise, the analysis applies as well to any other noise; the tariffs then need to be adjusted for $\mathbb{E}[\varepsilon]$).

(b) If the optimal tariff is not concave but still "smooth", it can be replicated with a family of sufficiently convex non-linear tariffs, such that the lower envelope of this family coincides with the original tariff – see Figure 1.7.

Consider for instance a menu of quadratic tariffs $\{\tau_\theta(q)\}_{\theta\in\Theta}$, where

$$\tau_\theta(q) = A(\theta) + p(\theta)q + bq^2.$$

When the agent chooses an output level q, the expected transfer is then

$$\begin{aligned}
\mathbb{E}[\tau_\theta(\tilde{q})] &= \mathbb{E}[\tau_\theta(q+\varepsilon)] = \mathbb{E}[A(\theta) + p(\theta)(q+\varepsilon) + b(q+\varepsilon)^2] \\
&= A(\theta) + p(\theta)q + \frac{b}{2}q^2(\theta) + b\sigma^2 \\
&= \tau_\theta(q) + b\sigma^2,
\end{aligned}$$

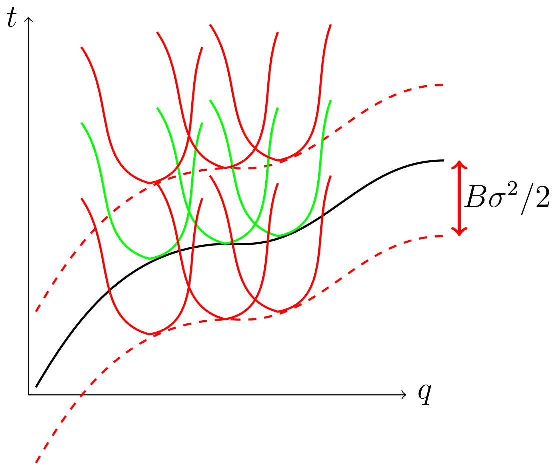

Figure 1.7: Replication of the non-concave optimal tariff with the family of convex non-linear tariffs

where σ^2 denotes the variance of the noise ε. But then, it suffices to reduce τ_θ by $b\sigma^2$ to ensure that the expected transfer coincides with the transfer that would be implemented in the absence of any noise. Therefore, the family of quadratic tariffs $\{\hat{\tau}_\theta(q) = \tau_\theta(q) - b\sigma^2\}_{\theta\in\Theta}$ achieves the second-best; in other words, as long as the first two moments of the distribution of the noise are known, then observing only the noisy version of the output does not effect the efficiency of the principal-agent relationship.

1.4.3 Interim Negotiation

Now suppose that agents could contract *before* they learn their types. While ex post incentive compatibility must still be satisfied, ex ante contracting replaces the ex post participation constraints (for each type of agent) with a unique, ex ante participation constraint. As the participation constraints are less demanding, the parties can enter into more efficient contracts. In particular, ex ante the principal can appropriate, through a lump sum transfer, the informational rents that the agent can secure ex post; intuitively, this reduces the cost of the informational rents, and may even make them costless, in which case the second-best contract will still involve first-best efficiency.

Achieving first-best efficiency

To see this, note first that both parties now aim to maximize their *expected* payoffs:

$$\max_{q(\cdot),t(\cdot)} \mathbb{E}[t(\theta) - c(q(\theta);\theta)]$$
$$\text{s.t.} \quad \forall\theta \in \Theta \quad , \begin{cases} \mathbb{E}[V(q(\theta);\theta) - t(\theta)] \geq 0, & (IR) \\ \theta \in \arg\max_{\tilde{\theta}} V(q(\tilde{\theta});\theta) - t(\tilde{\theta}). & (IC) \end{cases}$$

In a first-best world (thus ignoring (IC)), $q^{FB}(\theta)$ maximizes total surplus $W(q;\theta) = V(q;\theta) - C(q;\theta)$ and, under standard concavity assumptions, solves the first-order condition

$$\partial_q V(q;\theta) = \partial_q C(q;\theta).$$

For the sake of exposition, suppose that the first-best is implementable; that is, suppose there exists a profile of transfers $t(\theta)$, such that $(q^{FB}(\theta), t(\theta))$ is incentive-compatible. This will always be the case if for example the principal's payoff does not directly depend on the agent's type, that is, $C(q;\theta) = C(q)$:

- adopting the non-linear tariff $\hat{T}(q) = C(q)$ would then make the agent the residual claimant: he would thus maximize $V(q;\theta) - \hat{T}(q) = V(q;\theta) - C(q) = W(q;\theta)$ and thus choose $q^{FB}(\theta)$.
- alternatively, this could be achieved with the profile of transfers $\hat{t}(\theta) = \hat{T}\left(q^{FB}(\theta)\right)$; by construction, $q^{FB}(\theta) = \arg\max_q V(q;\theta) - \hat{T}(q)$ implies
$$\theta = \arg\max_{\tilde{\theta}} V\left(q^{FB}\left(\tilde{\theta}\right);\theta\right) - \hat{t}\left(\tilde{\theta}\right).$$

Let $r(\theta) = V(q^{FB}(\theta);\theta) - t(\theta)$ denote the rent that an agent of type θ would obtain under the profile $t(\theta)$ that implements the first-best, and denote by r the expected value of this rent,

$$r = \mathbb{E}[r(\theta)].$$

Ex ante, the principal has expected utility

$$\mathbb{E}[W^{FB}(\theta) - r(\theta)] = W^{FB} - r,$$

whereas the agent has expected utility r. Shifting the transfers uniformly (e.g., increasing $\hat{T}(q)$ by some amount, or all transfers $\hat{t}(\theta)$ by the same amount) does not affect incentive compatibility, and allows the principal to redistribute the surplus. In particular, the transfers

$$t^{FB}(\theta) \equiv t(\theta) + r,$$

allow the principal to appropriate ex-ante all the expected surplus (that is, the principal's ex ante utility is then equal to W^{FB}), leaving the agent with an ex-ante expected utility equal to 0.

Illustration: Price Discrimination

Let the payoff of the principal (seller) be given by $W(q,t) = t - cq$ and the payoff of the agent (buyer) be given by $U(q,t;\theta) = \theta V(q) - t$. The tariff $T(q) = cq$ would for sure lead to the first-best level q^{FB} but gives all the surplus to the buyer, leaving the seller with zero profit (whatever the buyer's type). To maximize the seller's profit, it suffices to adjust this tariff by an amount F,

$$T^{FB}(q) \equiv F^{FB} + cq,$$

where

$$F^{FB} \equiv \mathbb{E}\left[\theta V(q^{FB}(\theta)) - c(q^{FB}(\theta))\right] = W^{FB}.$$

Under the tariff T^{FB}, the principal appropriates all the surplus.

Risk Aversion

Thus far, we have relied on the assumption that all parties are risk-neutral. We now relax that assumption and consider the two polar cases where either the principal or the agent is risk-averse.

Risk-Averse Principal Suppose that, while the agent's utility is still given by $V(q;\theta) - t$ (where $V'' < 0$), the principal now has an objective function given by $U_P(t - C(q))$, where $C'' > 0$ and $U_P'' < 0 < U_P'$. The first-best level of trade $q^{FB}(\theta)$ satisfies $\partial_q V(q;\theta) = C'(q)$ and is implementable via the tariff

$$\hat{T}(q) = C(q).$$

This leads indeed the agent to choose the right level of trade, $q^{FB}(\theta)$, and perfectly insures the principal against the risk about the agent's type, but it also gives all the surplus to the agent. As above, an optimal tariff for the principal then simply consists in shifting $\hat{T}$ up by the amount W^{FB}:

$$T^{FB}(q) = W^{FB} + C(q).$$

With this tariff, ex post the principal obtains W^{FB} regardless of the agent' type θ; there is thus again perfect insurance: it is the agent who ex post bears all the risk of the particular draw of θ.

Remark. This no longer works if the principal's objective is directly affected by the agent's type, i.e. if the principal's cost function takes the form $C(q;\theta)$; in that case, the transfers

$$\hat{t}(\theta) = C(q^{FB}(\theta);\theta)$$

would no longer be incentive-compatible: the agent would seek to maximize

$$\varphi\left(\tilde{\theta},\theta\right) = V(q(\tilde{\theta});\theta) - C(q^{FB}(\tilde{\theta});\tilde{\theta}),$$

but then

$$\partial_{\tilde{\theta}}\varphi|_{\tilde{\theta}=\theta} = \partial_{\tilde{\theta}}C(q^{FB}(\theta);\theta) \neq 0;$$

that is, the agent has an incentive to deviate.

Risk-Averse Agent Suppose now that, while the principal's utility is given as before by $t-C(q;\theta)$, the agent's utility is now given by $U_A(V(q;\theta)-t)$, where $V'' < 0$ and $U_A'' < 0 < U_A'$. In a first-best world, the principal should therefore fully insure the agent:

$$t(\theta) = V(q(\theta);\theta),$$

but this is not incentive compatible: the agent would seek to maximize his utility:

$$\max_{\tilde{\theta}} \varphi\left(\tilde{\theta},\theta\right) = V(q(\tilde{\theta});\theta) - V(q(\tilde{\theta});\tilde{\theta}),$$

where

$$\partial_{\tilde{\theta}}\varphi|_{\tilde{\theta}=\theta} = -\partial_{\theta}V(q(\theta);\theta) \neq 0.$$

There is thus again a trade off between insurance and rent extraction.

Example. Suppose that the principal's utility is $t-C\left(q\right)$ and the agent's utility is given by $U(\theta q - t)$, where θ can take on one of two values, $\underline{\theta}$ or $\bar{\theta}$, with probabilities $\underline{\mu}$ and $\bar{\mu} = 1-\underline{\mu}$ respectively. Assume that $U(0) = 0$ and $U'' < 0 < U'$. The principal's maximization problem can be written as

$$\begin{aligned}
\max_{\underline{q},\underline{t},\bar{q},\bar{t}} & \; W = \underline{\mu}\left[\underline{t} - C(\underline{q})\right] + \bar{\mu}\left[\bar{t} - C(\bar{q})\right] \\
\text{s.t.} \quad & \underline{\mu}U(\underline{\theta}\underline{q} - \underline{t}) + \bar{\mu}U(\bar{\theta}\bar{q} - \bar{t}) \geq 0, \\
& \bar{\theta}\bar{q} - \bar{t} \geq \bar{\theta}\underline{q} - \underline{t}, \\
& \underline{\theta}\underline{q} - \underline{t} \geq \underline{\theta}\bar{q} - \bar{t}.
\end{aligned}$$

The incentive-compatibility conditions can be rewritten as

$$\begin{aligned}\bar{r} &\geq \underline{r} + (\bar{\theta} - \underline{\theta})\underline{q} \\ \underline{r} &\geq \bar{r} - (\bar{\theta} - \underline{\theta})\bar{q},\end{aligned}$$

which we combine into the single condition

$$(\bar{\theta} - \underline{\theta})\bar{q} \geq \bar{r} - \underline{r} \geq (\bar{\theta} - \underline{\theta})\underline{q}.$$

In the first-best (i.e., absent incentive issues), the agent would be fully insured: $\bar{r} = \underline{r}$; therefore, it is the lower (i.e. right) IC constraint that is the binding one. Thus, we can rewrite the principal's problem as

$$\begin{aligned}\max_{\underline{q},\underline{t},\bar{q},\bar{t}} W &= \underline{\mu}(\underline{\theta}\underline{q} - C(\underline{q}) - \underline{r}) + \bar{\mu}(\bar{\theta}\bar{q} - C(\bar{q}) - \bar{r}) \\ \text{s.t.} \quad & \underline{\mu}U(\underline{r}) + \bar{\mu}U(\bar{r}) \geq 0 \\ & \bar{r} - \underline{r} \geq (\bar{\theta} - \underline{\theta})\underline{q}.\end{aligned}$$

Assign the two constraints the Lagrange multipliers λ and ν, respectively. The Lagrangian is

$$L = W + \lambda\left[\underline{\mu}U(\underline{r}) + \bar{\mu}U(\bar{r})\right] + \nu\left[\bar{r} - \underline{r} - (\bar{\theta} - \underline{\theta})\underline{q}\right],$$

and the first-order conditions are

$$\begin{array}{llcl}\text{w.r.t. } \bar{q}: & C'(q) = \bar{\theta} & \Rightarrow & \bar{q} = q^{FB}(\bar{\theta}), \\ \text{w.r.t. } \underline{q}: & C'(q) = \underline{\theta} - \frac{\nu}{\underline{\mu}}(\bar{\theta} - \underline{\theta}) & \Rightarrow & \underline{q} < q^{FB}(\underline{\theta}), \\ \text{w.r.t. } \bar{r}: & -\bar{\mu} + \lambda\bar{\mu}U'(\bar{r}) + \nu = 0 & \Rightarrow & U'(\bar{r}) = \dfrac{1}{\lambda} - \dfrac{\nu}{\lambda\bar{\mu}}, \\ \text{w.r.t. } \underline{r}: & -\underline{\mu} + \lambda\underline{\mu}U'(\underline{r}) - \nu = 0 & \Rightarrow & U'(\underline{r}) = \dfrac{1}{\lambda} + \dfrac{\nu}{\lambda\underline{\mu}}.\end{array}$$

The Lagrange multipliers ν and λ are positive (Lagrange multipliers are non-negative by construction, and both constraints are necessarily binding at the optimum), $U'(\bar{r}) < U'(\underline{r})$. Since $U'' < 0$ by assumption, it follows that $\bar{r} > 0 > \underline{r}$; there is thus *partial insurance.*

1.4.4 Countervailing Incentives

Suppose now that, while the Spence-Mirrlees condition holds with a positive sign, either $\partial_\theta V < 0$ or the agent's reservation level of utility depends on the type, $\underline{U}(\theta)$, and increases faster than the informational rent $r^{SB}(\theta)$ characterized above. In that case other individual-rationality constraints (i.e., for other types than the lowest one) may be binding. This, in turn, may affect which incentive-compatibility constraints are relevant.

Example. Let the agent's utility be given by $U = \theta q - t$, where the agent's type θ can take on one of two values $\underline{\theta}$ or $\bar{\theta}$, with probabilities $\underline{\mu}$ and $\bar{\mu}$, respectively. Assume $\bar{\theta} > \underline{\theta}$, and denote the reservation utilities of each type by $\underline{U}$ and $\bar{U}$, respectively. Assume that the SM_+holds, and that $\partial_\theta V > 0$. Finally, define $\Delta U = \bar{U} - \underline{U}$; we will distinguish five cases, according to the value of ΔU.

Case 1: $\Delta U = 0$, i.e. $\bar{U} = \underline{U}$. In that case, the analysis follows the same steps as above, the only difference being that the agent's rent is uniformly increased by $\bar{U} = \underline{U}$:

- For the high-type agent, $\bar{q} = q^{FB}(\bar{\theta})$, such that $C'(q) = \bar{\theta}$: there is no distortion at the top.
- For the low-type agent, $\underline{q} = \hat{q} < q^{FB}(\underline{\theta})$, such that $C'(\hat{q}) = \underline{\theta} - \frac{\underline{\mu}}{\bar{\mu}}(\bar{\theta} - \underline{\theta})$.
- The rents are given by $\underline{r} = \underline{U}$ and $\bar{r} = \underline{r} + (\bar{\theta} - \underline{\theta})\hat{q}$, which satisfies $\bar{r} > \bar{U}$.

In summary,

$$\begin{aligned} &\underline{q} = \hat{q} < q^{FB}(\underline{\theta}), \underline{r} = \underline{U}, \\ &\bar{q} = q^{FB}(\bar{\theta}), \qquad \bar{r} = \underline{r} + (\bar{\theta} - \underline{\theta})\hat{q} > \bar{U}. \end{aligned}$$

Case 2: $\Delta U \leq (\bar{\theta} - \underline{\theta})\hat{q}$. The above solution – which is the solution to a relaxed problem, where the low-type agent's participation constraint $(\underline{IR})$ and the high-type agent's incentive constraint $\left(\overline{IC}\right)$ are ignored – still satisfies the omitted constraints; in particular, $\underline{r} = \underline{U}$ and $\bar{r} = \underline{r} + (\bar{\theta} - \underline{\theta})\hat{q}$ together imply $\bar{r} > \bar{U}$. Thus the solution remains the same as in Case 1.

Case 3: $(\bar{\theta} - \underline{\theta})\hat{q} < \Delta U < (\bar{\theta} - \underline{\theta})q^{FB}(\underline{\theta})$. There are three binding constraints in this case, as the high-type participation constraint $\left(\overline{IR}\right)$ starts binding; that is, setting $\underline{r} = \underline{U}$ and $\bar{r} = \underline{r} + (\bar{\theta} - \underline{\theta})\hat{q}$ as above would violate $\left(\overline{IR}\right)$, since $\underline{r} + (\bar{\theta} - \underline{\theta})\hat{q} < \bar{U}$. In the absence of this additional constraint $\left(\overline{IR}\right)$, it would be optimal to trade at the first-best level with a high-type agent $\left(\bar{q} = q^{FB}(\bar{\theta})\right)$ but distort the level of trade assigned to a low type $\underline{\theta}$, $\underline{q}$, down to $\hat{q}$, in order to reduce the rent left to a high type $\bar{\theta}$, $\bar{r}$; but since a larger rent needs to be left anyway to meet the reservation level $\bar{U}$ of a high type $\bar{\theta}$, there is no point distorting $\underline{q}$ down to $\hat{q}$: it suffices to reduce $\underline{q}$ to the level $\underline{q}^{SB}$ that generates the "appropriate" utility differential, namely, such that $(\bar{\theta}-\underline{\theta})\underline{q}=\Delta U$; we obtain $\underline{r}=\underline{U}$ and $\bar{r}=\bar{U}$.

$$\begin{aligned} \hat{q}<\underline{q}=\underline{q}^{SB}<q^{FB}(\underline{\theta}), &\quad \underline{r}=\underline{U}, \\ \bar{q}=q^{FB}(\bar{\theta}), &\quad \bar{r}=\bar{U}. \end{aligned}$$

Case 4: $(\bar{\theta}-\underline{\theta})q^{FB}(\underline{\theta}) \leq \Delta U \leq (\bar{\theta}-\underline{\theta})q^{FB}(\bar{\theta})$. In this case, only the participation constraints matter: as the rents needed to meet both types' participation constraints make the first-best incentive compatible, the incentive constraints can be ignored and the second-best coincides with the first-best:

$$\begin{aligned} \underline{q}=q^{FB}(\underline{\theta}), &\quad \underline{r}=\underline{U}, \\ \bar{q}=q^{FB}(\bar{\theta}), &\quad \bar{r}=\bar{U}. \end{aligned}$$

Case 5: $(\bar{\theta}-\underline{\theta})q^{FB}(\bar{\theta}) < \Delta U$. In this case, the participation constraint of the high type is binding and incentive-compatibility constraint for the low type, namely,

$$\underline{r} \geq \bar{r}-(\bar{\theta}-\underline{\theta})\bar{q},$$

becomes binding. If we ignore the other two constraints, it would be optimal to set $\underline{q}$ to the first-best level $\left(\underline{q}=q^{FB}(\underline{\theta})\right)$ and to distort $\bar{q}$ *upwards* so as to reduce the rent $\underline{r}$ left to a low type $\underline{\theta}$: that is, maximizing

$$\max_{\bar{q},\underline{q}} \bar{\mu}\left[\bar{\theta}-\bar{q}-c(\bar{q})-\underline{r}\right]+\underline{\mu}\left[\underline{\theta}\underline{q}-c(\underline{q})-\bar{r}+(\bar{\theta}-\underline{\theta})\bar{q}\right],$$

would lead to $\underline{q}=q^{FB}(\underline{\theta})$ and

$$\bar{q}=\tilde{q}\equiv\bar{\theta}+\frac{\underline{\mu}}{\bar{\mu}}(\bar{\theta}-\underline{\theta})>q^{FB}(\bar{\theta}).$$

We can then distinguish two subcases:

i) If $\Delta U<(\bar{\theta}-\underline{\theta})\tilde{q}$, then distorting $\bar{q}$ up to $\tilde{q}$ is excessive, as it would generate a higher rent differential than is needed to meet the participation constraints of the two types; in that case, it suffices to distort $\bar{q}$ to the level just needed to accommodate the utility differential: that is, $\bar{r}=\bar{U}$ and $\bar{q}=\bar{q}^{SB}$ such that $(\bar{\theta}-\underline{\theta})\bar{q}=\Delta U$, so as to accommodate $\underline{r}=\underline{U}$. This subcase is thus the mirror image of Case 3, in which there are three binding constraints: both individual rationality constraints, plus the incentive constraint of the low type. We have:

$$\underline{q} = q^{FB}(\underline{\theta}), \qquad \underline{r} = \underline{U},$$
$$\tilde{q} > \bar{q} = \bar{q}^{SB} > q^{FB}(\bar{\theta}), \bar{r} = \bar{U}.$$

ii) If instead $\Delta U \geq (\bar{\theta} - \underline{\theta})\tilde{q}$, then the solution to the relaxed problem, where we ignore the high-type agent's participation constraint $\left(\overline{IR}\right)$ and the low-type agent's incentive constraint $(\underline{IC})$, which is given by $\underline{q} = q^{FB}(\underline{\theta})$ and $\bar{q} = \tilde{q} > q^{FB}(\bar{\theta})$, so $\bar{r} = \bar{U}$ and $\underline{r} = \bar{U} - (\bar{\theta} - \underline{\theta})\tilde{q}$, satisfy the omitted constraints; in particular, $\bar{r} = \bar{U}$ and $\underline{r} = \bar{U} - (\bar{\theta} - \underline{\theta})\tilde{q}$ together imply $\underline{r} > \underline{U}$. This subcase is thus the mirror image of Case 2, in which two constraints are binding: the individual rationality constraint of the high type and the incentive constraint of the low type. We have:

$$\underline{q} = q^{FB}(\underline{\theta}), \qquad \underline{r} = \underline{U},$$
$$\bar{q} = \tilde{q} > q^{FB}(\bar{\theta}), \bar{r} = \bar{U}.$$

Remark: The second-best is continuous with respect to the utility differential ΔU (see Figure).

1.4.5 Stochastic Contracts

Suppose that the principal and agent consider contracting on lotteries of the form $(\tilde{q}, \tilde{t})$, where $\tilde{q}$ and $\tilde{t}$ can now depend on the realization of some random variable. The principal then seeks to maximize her expected utility, $\mathbb{E}[\tilde{t} - C(\tilde{q}; \theta)]$, whereas the agent maximizes his expected utility, $\mathbb{E}[V(\tilde{q}; \theta) - \tilde{t}]$. Both parties are risk-neutral with respect to transfers, and thus any lottery on $\tilde{t}$ is formally equivalent to a deterministic transfer equal to the expected value $\mathbb{E}\left[\tilde{t}\right]$. Furthermore, if C is convex in q (i.e., $\partial^2_{q^2} C(\cdot) > 0$), then exposing the principal to random shocks on q is not efficient; the same applies to the agent if V is concave in q (i.e., $V''(\cdot) < 0$). In such a case, under complete information there is thus no role for lotteries, as both agents would prefer a deterministic contract: replacing a lottery $\left(\tilde{t}, \tilde{q}\right)$ with for example the deterministic contract $\left(\mathbb{E}[\tilde{t}], \mathbb{E}[\tilde{q}]\right)$ would benefit both parties. Yet, under incomplete information, lotteries can be helpful if they relaxing incentive constraints.

To see this, define $\varphi(\tilde{\theta}, \theta) = \mathbb{E}[V(q(\tilde{\theta}); \theta) - t(\tilde{\theta})]$; a lottery may then help if it the expression of φ is concave in $q\left(\tilde{\theta}\right)$, as it reduces the potential benefit from cheating.

Example. Suppose that the two parties' objectives are respectively given by $U = V(q;\theta) - t$ and $W = t - cq$. The principal's program can be written as:

$$\begin{aligned} \max \quad & \underline{\mu}\mathbb{E}\left[\underline{\tilde{t}} - c\underline{\tilde{q}}\right] + \bar{\mu}\mathbb{E}\left[\overline{\tilde{t}} - c\overline{\tilde{q}}\right] \\ & (\underline{IR}) : \mathbb{E}\left[V(\underline{\tilde{q}};\underline{\theta}) - \underline{\tilde{t}}\right] \geq 0, \\ & \left(\overline{IC}\right) : \mathbb{E}\left[V(\overline{\tilde{q}};\bar{\theta}) - \overline{\tilde{t}}\right] \geq \mathbb{E}\left[V(\underline{\tilde{q}};\bar{\theta}) - \underline{\tilde{t}}\right], \end{aligned}$$

or, equivalently, in terms of rents $r(.)$:

$$\begin{aligned} \max \quad & \underline{\mu}\mathbb{E}\left[V(\underline{\tilde{q}};\underline{\theta}) - c\underline{\tilde{q}} - \underline{\tilde{r}}\right] + \bar{\mu}\mathbb{E}\left[V(\overline{\tilde{q}};\bar{\theta}) - c\overline{\tilde{q}} - \overline{\tilde{r}}\right] \\ & (\underline{IR}) : \mathbb{E}\left[\underline{\tilde{r}}\right] \geq 0, \\ & \left(\overline{IC}\right) : \mathbb{E}\left[\overline{\tilde{r}}\right] \geq \mathbb{E}\left[\underline{\tilde{r}}\right] + \mathbb{E}\left[\phi(\underline{\tilde{q}})\right]. \end{aligned}$$

where

$$\phi(\underline{\tilde{q}}) \equiv V(\underline{\tilde{q}};\bar{\theta}) - V(\underline{\tilde{q}};\underline{\theta}).$$

If ϕ is convex in $q$there is no scope for lotteries. However if ϕ is concave $\mathbb{E}(\phi(\underline{\tilde{q}}))<\phi(\mathbb{E}\left[\underline{\tilde{q}}\right])$, in which case opting for a lottery for $\underline{q}$ may relax the high-type agent's incentive constraint.

To see this, suppose first that the parties restrict attention to deterministic contracts. The principal will thus set $\underline{r} = 0$ and $\bar{r} = \phi\left(\underline{q}\right)$, $\bar{q} = q^{FB}\left(\bar{\theta}\right)$ and then choose $\underline{q}$ so as to maximize

$$\hat{W}(\underline{q};\underline{\theta}) \equiv V(\underline{q};\underline{\theta}) - c\underline{q} - \frac{\bar{\mu}}{\underline{\mu}}\phi(\underline{q}).$$

But then, if the virtual objective $\hat{W}(\underline{q};\underline{\theta})$ has several optima in $\underline{q}$ (which is indeed possible if ϕ is concave in $\underline{q}$), then replacing any deterministic solution with a lottery over these various solutions would (i) not affect the principal's objective (since by construction it is constant over these solutions), but (ii) relax the high-type agent's incentive constraint $\left(\overline{IC}\right)$; the principal could then improve over deterministic contracts by reducing $\overline{r}$.

1.4.6 Dynamics

Suppose now that there are two periods, $\tau = 1, 2$; in each period τ, the parties' payoffs are as before given by $t_\tau - C(q_\tau, \theta)$ for the principal and $V(q_\tau, \theta) - t_\tau$ for the agent (note that, by assumption, the agent's type is constant over time). The two parties maximize the (expected) sum of their discounted payoffs, using the same discount factor, $\delta = \frac{1}{1+r}$.

The principal's program can thus be written as:

$$\begin{array}{lll} \max\limits_{q_1,t_1,q_2,t_2} & & \mathbb{E}[t_1(\theta) - C(q_1(\theta);\theta)] + \delta\mathbb{E}[t_2(\theta) - C(q_2(\theta);\theta)] \\ \text{s.t. } \forall\theta \in \Theta, & (IR_\theta): & V(q_1(\theta);\theta) - t_1(\theta) + \delta[V(q_2(\theta);\theta) - t_2(\theta)] \geq 0 \\ & (IC_\theta): & \theta \in \arg\max\limits_{\widetilde{\theta}} V(q_1(\tilde{\theta});\theta) - t_1(\tilde{\theta}) + \delta[V(q_2(\tilde{\theta});\theta) - t_2(\tilde{\theta})] \end{array}$$

Dividing all payoffs by$1 + \delta$ leads to:

$$\begin{array}{lll} \max\limits_{q_1,t_1,q_2,t_2} & & \frac{1}{1+\delta}\mathbb{E}[t_1(\theta) - C(q_1(\theta);\theta)] + \frac{\delta}{1+\delta}\mathbb{E}[t_2(\theta) - C(q_2(\theta);\theta)] \\ \text{s.t. } \forall\theta \in \Theta, & (IR_\theta): & \frac{1}{1+\delta}V(q_1(\theta);\theta) - t_1(\theta) + \frac{\delta}{1+\delta}[V(q_2(\theta);\theta) - t_2(\theta)] \geq 0 \\ & (IC_\theta): & \theta \in \arg\max\limits_{\widetilde{\theta}} \frac{1}{1+\delta}V(q_1(\tilde{\theta});\theta) - t_1(\tilde{\theta}) + \frac{\delta}{1+\delta}[V(q_2(\tilde{\theta});\theta) - t_2(\tilde{\theta})] \end{array}$$

This program is now formally identical to the program that, in a static (i.e., one-period) context, the principal would be restricted to use a stochastic contract, leading to $(q_1(\theta), t_1(\theta))$ with probability $\frac{1}{1+\delta}$ and to $(q_2(\theta), t_2(\theta))$ with probability $\frac{\delta}{1+\delta}$. Therefore:

- If in the static context the optimal contract is deterministic, implying that it would be optimal to choose a "degenerate" stochastic contract where $(q_1(\theta), t_1(\theta)) = (q_2(\theta), t_2(\theta))$, then in the dynamic context it is optimal to have a stationary contract, with the same terms in both periods.

- The argument carries over to situations in which it would be optimal to rely on lotteries in the static context: it is always possible to "add" to the lottery a second step in which a binary random variable would be drawn (with probabilities $\frac{1}{1+\delta}$ and $\frac{\delta}{1+\delta}$), and then implement the same outcome of the lottery, whatever that of the binary variable; likewise, in the dynamic context it would be optimal to rely on the static lottery, and then stick to the realization of this lottery over the two periods.

Remark: independent types. The above analysis relies on the assumption that the agent's type is fixed once and for all. In the other polar case in which it is drawn independently in each period then, at the beginning of the second period, the principal and the agent would have symmetric information about θ_2, the agent's type in period 2. The same logic as above would then lead the principal to offer a contract replicating the optimal static contract with ex post negotiation for the first period, and the optimal static contract with *interim* negotiation for the second period.

Remark: commitment. The above analysis relies also heavily on the parties' commitment abilities. For example, the principal commits to pay "for ever" a rent to high-type agents, but would have an incentive to renege on her promise in later periods. Likewise, the parties commit themselves to stick "for ever" to inefficient levels of trade if the agent turns out to be of a low type, but once the agent has revealed his type in the first period, then in the subsequent periods the parties would have a joint interest to replace the original, inefficient contract with a more efficient one (sharing the additional gains from trade). We briefly discuss below these two issues in turn. Before that, we note here that, in the absence of commitment, the revelation principle does not apply: while a direct mechanism would require the agent to reveal all his information at the beginning of the first period, it may be desirable to have this information revealed only progressively. Yet, Bester & Strausz (*Econometrica* 2001) offer a modified version of the revelation principle and show that, without loss of generality, the parties can restrict attention to contracts:

- that offer as many options as there are agent types;
- and such that a given type θ picks the option designed for that type with positive probability.

Unilateral renegotiation. Suppose that the parties cannot commit themselves beyond the current period (spot contracting). This implies that any rent promised to a high-type agent has to be paid in the first period; but this, in turn, may lead a low-type agent to pretend having a high type, in order to pocket the associated rent, and then reject any contract in the subsequent periods ("hit-and-run" strategies). As a result, it may be impossible to pay the rent needed to induce truthful revelation in the first period (this phenomenon is referred to as the "ratchet effect": the inability to pay out informational rents leads high-type agents to behave as low-types ones).

Example: Suppose that the agent's utility is given by $\theta q - t$ and that there are two types, $\underline{\theta}$ and $\overline{\theta} > \underline{\theta}$, and T periods; any contract that induces the agent to reveal his type in the first period will lead to efficient trade in the following periods; thus, the rent from mimicking a low type would be

$$R = (\overline{\theta} - \underline{\theta})[\underline{q}_1 + (\delta ... + \delta^{T-1})\underline{q}^{FB}] = (\overline{\theta} - \underline{\theta})\left(\underline{q}_1 + \delta \frac{1-\delta^T}{1-\delta}\underline{q}^{FB}\right).$$

But since the principal cannot credibly commit to pay any rent in the future, the whole rent has to be paid in period 1, which implies that the transfer to a high-type must be equal to

$$\overline{t}_1 = \overline{\theta}\overline{q} - R.$$

As $T \to \infty$, and $\delta \to 0$, $R \to \infty$ and thus $\bar{t}_1 \to -\infty$; but then, a low-type agent would have an incentive to pretend having a high-type, so as to benefit from $\bar{t}_1$ – and this, regardless of the actual levels of trade $\underline{q}_1$ and $\bar{q}_1$, as long as they differ from each other: while the agent is then expected to trade $\overline{q}^{FB}$ at the "full" price $\bar{\theta}\overline{q}^{FB}$ in each of the following period, a low-type agent that cheats in the first period is then free to refuse any trade later on, which makes the cheating deviation profitable. In other words, for T and/or δ large enough, it is not possible to have the agent's type fully revealed in the first period.

Bilateral renegotiation. Suppose now that the parties can commit over future periods, but are unable to commit not to renegotiate the original contract. Under full commitment, solving the rent vs. efficiency trade-off would lead the principal to distort the levels of trade offered to low types (e.g., $\underline{q}^{SB} < \underline{q}^{FB}$ in the above two-type example). But now, once the agent has revealed his type it is no longer possible to stick to inefficient levels in the future periods: the parties would then replace the inefficient level of trade $\left(\underline{q}^{SB}\right)$ with the efficient one $\left(\underline{q}^{FB}\right)$. The implication is that revealing the agent's type is more costly, as then the rent that has to be paid is based on the higher, efficient level of trade for the future periods; this, in turn, may reduce the pace at which it is optimal to have the information revealed.

Example: Consider the same two-type example as above, and suppose that, in the first period:

- type $\underline{\theta}$ chooses $\underline{q}_1$ with probability 1;
- type $\bar{\theta}$ choose $\bar{q}_1 = \bar{q}^{FB}$ with probability $1 - \nu$ and $\underline{q}_1$ with probability ν.

If the principal observes $\overline{q}^{FB}$ in period 1, then he knows for sure that the agent has a high type; in the second period, the quantity will thus be $\overline{q}_2 = \overline{q}^{FB}$; by contrast, observing $\underline{q}_1$ does not fully reveal the agent's type: the principal's revised belief are then that the agent is of high type with probability $\overline{\mu}_2 = \frac{\bar{\mu}(1-\nu)}{\bar{\mu}(1-\nu)+\underline{\mu}}$, leading to $\underline{q}_2(\nu)$ such that:

$$C'(\underline{q}_2) = \underline{\theta} - \frac{\bar{\mu}_2}{\underline{\mu}_2}\left(\bar{\theta} - \underline{\theta}\right) = \underline{\theta} - \nu\frac{\bar{\mu}}{\underline{\mu}}\left(\bar{\theta} - \underline{\theta}\right).$$

If $\nu = 0$ (in which case the agent fully reveal his type in period 1), then $\underline{q}_2$ is efficient: $\underline{q}_2 = \underline{q}^{FB}$; however, by only partially revealing his type (i.e., by randomizing between the two levels of trade in period 1), the agent entertains some ambiguity in the second period, which allows the parties to credibly commit to maintaining a lower level of trade (and the more so, the larger ν), which in turn allows the principal to reduce the total rent that needs to be paid to the agent, equal to $(\bar{\theta} - \underline{\theta}) \left(\underline{q}_1 + \delta \underline{q}_2 \underline{q}_2\right)$.

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