# Lecture 2: Neoclassical Growth Theory (*Acemoglu 2009*, Chapter 8) adapted from Fabrizio Zilibotti

Kjetil Storesletten

September 1, 2014

# Introduction

- Ramsey or Cass-Koopmans model:
  differs from the Solow model insofar as it explicitly
  models the consumer side and endogenizes savings
- Beyond its use as a basic growth model,
  it is also a workhorse for many areas of macroeconomics
- Example: real and monetary business cycle theory

# Preferences, Technology and Demographics I

- Infinite-horizon, continuous time.
- Representative household with instantaneous utility function

$$u(c(t)),$$

Assumption $u(c)$ is strictly increasing, concave, twice continuously differentiable with derivatives $u'$ and $u''$, and satisfies the following Inada type assumptions:

$$\lim_{c \to 0} u'(c) = \infty \text{ and } \lim_{c \to \infty} u'(c) = 0.$$

- Suppose representative household represents set of identical households (normalized to 1)
- Each household has an instantaneous utility function $u(c(t))$
- $L(0) = 1$ and

$$L(t) = \exp(nt)$$

# Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically
- Objective function of the representative household at $t = 0$:

$$U(0) \equiv \int_0^\infty \exp(-\rho t) \cdot L(t) \cdot u(c(t))\, dt \qquad (1)$$
$$= \int_0^\infty \exp(-(\rho - n)t) \cdot u(c(t))\, dt,$$

  where
  - $c(t)$=consumption per capita at $t$,
  - $\rho$=subjective discount rate, so that effective discount rate is $\rho - n$.
- Objective function (1) embeds:
  - Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively
  - Strict concavity of $U(\cdot)$
- Thus each household member has an equal consumption, $c(t) \equiv \frac{C(t)}{L(t)}$

# Preferences, Technology and Demographics III

ASSUMPTION: $\rho > n$

- Benchmark model without any technological progress
- Factor and product markets are competitive
- Production possibilities set of the economy is represented by

$$Y(t) = F[K(t), L(t)],$$

- F features constant returns to scale and Inada conditions, i.e., $Y = F_K \cdot K + F_L \cdot L$ (Euler Theorem) and $\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = \infty, \qquad \lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = 0$

# Preferences, Technology and Demographics IV

- Define variables in p.c. terms, $x(t) \equiv X(t)/L(t)$
- Per capita production function $f(\cdot)$

$$y(t) = F\left[\frac{K(t)}{L(t)}, 1\right] \equiv f(k(t)),$$

- Competitive factor markets imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)).$$

and (from the Euler theorem)

$$\begin{aligned} w(t) &= F_L[K(t), L(t)] = \frac{F[K(t), L(t)]}{L(t)} - F_K[K(t), L(t)]\frac{K(t)}{L(t)} \\ &= f(k(t)) - k(t)f'(k(t)). \end{aligned}$$

# Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time $t$ by $\mathcal{A}(t)$. Then,

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + w(t)L(t) - c(t)L(t)$$

- $r(t)$ is the market flow rate of return on assets, and $w(t)L(t)$ is the flow of labor income earnings of the household.
- Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

- Household assets can consist of capital stock, $K(t)$, which they rent to firms and bonds in zero net supply, $B(t)$.

# Preferences, Technology and Demographics VI

- Given no uncertainty, arbitrage implies that the rate of return on bonds must equal the net return on capital (after depreciation at the rate $\delta$).
- Both returns must equal $r(t) \Rightarrow$

$$r(t) = R(t) - \delta$$

- Moreover, market clearing $\Rightarrow$

$$a(t) = k(t)$$

# The Budget Constraint I

- Let us return to the flow (or dynamic) budget constraint

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t)$$

- Imposing that the flow constraint holds for all $t \in [0, \infty[$ is not sufficient to ensure that a proper budget constraint hold unless we impose a lower bound on assets
- A dynasty could increase its consumption by running an ever growing debt

# The Budget Constraint II

- No-Ponzi Game Condition
- Total debt cannot grow at a rate exceeding the interest rate;

$$\lim_{t \to \infty} \mathcal{A}(t) \exp\left(-\int_0^t r(s)\, ds\right) \geq 0.$$

- Equivalently, debt per capita cannot grow at a rate higher than $r - n$ :

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) \geq 0.$$

- Since it will never be optimal to have positive wealth asymptotically (formally, this will be captured by a Transversality Condition, TVC) the no-Ponzi-game condition can in fact be strengthened to:

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) = 0.$$

# The Budget Constraint III

- The no-Ponzi Game rules out the possibility for agents to borrow to finance present consumption and then use future borrowings to roll over the debt and pay the interest
- It can be shown formally (see textbook for a proof) that the no-Ponzi-game condition + period budget constraint ensures that the individual's lifetime budget constraint holds in infinite horizon

$$\begin{aligned} & \int_0^\infty c(t) \exp\left(-\int_0^t (r(s)-n)\, ds\right) dt \\ = \quad & a(0) + \int_0^\infty w(t) \exp\left(-\int_0^t (r(s)-n)\, ds\right) dt \end{aligned}$$

# Definition of Equilibrium

**Definition** A competitive equilibrium of the Ramsey economy consists of paths $[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock $K(0)$ and the time path of prices $[w(t), R(t)]_{t=0}^{\infty}$, and all markets clear.

- Notice:
  the definition refers to the entire path of quantities and prices, not just steady-state equilibrium.

# Household Maximization I

- Set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t)[w(t) + (r(t) - n)a(t) - c(t)],$$

- The solution must satisfy

$$\begin{aligned} (1) \quad \text{FOC} \quad : \quad & \hat{H}_c(a, c, \mu) = 0 \\ \Leftrightarrow \quad & u'(c(t)) = \mu(t) \end{aligned}$$

$$\begin{aligned} (2)\text{EE} \quad : \quad & \hat{H}_a(a, c, \mu) = -\dot{\mu}(t) + (\rho - n)\mu(t) = (r(t) - n)\mu(t) \\ \Leftrightarrow \quad & \frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho) \end{aligned}$$

$$(3) \quad \text{BC:} \quad \dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

$$(4) \quad \text{TVC:} \quad \lim_{t \to \infty} [\exp(-(\rho - n)t) \cdot \mu(t) \cdot a(t)] = 0$$

# Household Maximization II

- Take logarithms in the FOC and differentiate with respect to time

$$\frac{u''(c(t))\,c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

- Substituting into EE, obtain another form of the consumer Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t)-\rho)$$

where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))\,c(t)}{u'(c(t))}$$

is the elasticity of the marginal utility $u'(c(t))$.

- Consumption will grow over time when the discount rate is less than the rate of return on assets.

# Household Maximization III

- Speed at which consumption will grow is related to the IES, elasticity of marginal utility of consumption, $\varepsilon_u(c(t))$.
- $\varepsilon_u(c(t))$ can also be interpreted (see book) as the inverse of the *intertemporal elasticity of substitution* (IES):
  - regulates willingness to substitute consumption over time.
- Suppose

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c & \text{if } \theta = 1 \end{cases},$$

- This utility function (CRRA) induces a constant IES. In particular, $\varepsilon_u(c(t)) = \theta$, so $1/\theta$ is the constant IES.
- CRRA is necessary to have balanced growth.

# Household Maximization IV

- Under CRRA utility,

$$\mu(t) = c(t)^{-\theta}$$

and the consumer Euler equation yields:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho) = -\theta \frac{\dot{c}(t)}{c(t)} \Longrightarrow \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta}$$

- Thus, integrating,

$$\mu(t) = \mu(0) \exp\left(-\int_0^t (r(s) - \rho)\, ds\right),$$

# Household Maximization V

- Consider the TVC

$$
\begin{aligned}
0 &= \lim_{t\to\infty}\left[\exp\left(-\left(\rho-n\right)t\right)\cdot a\left(t\right)\cdot\mu\left(t\right)\right] \\
&= \lim_{t\to\infty}\left[\exp\left(-\left(\rho-n\right)t\right)\cdot a\left(t\right)\cdot\mu\left(0\right)\exp\left(-\int_0^t\left(r\left(s\right)-\rho\right)ds\right)\right] \\
&= \lim_{t\to\infty}\left[a\left(t\right)\exp\left(-\int_0^t\left(r\left(s\right)-n\right)ds\right)\right]\cdot c\left(0\right)^{-\theta}.
\end{aligned}
$$

- Thus, $\lim_{t\to\infty}\left[a\left(t\right)\exp\left(-\int_0^t\left(r\left(s\right)-n\right)ds\right)\right]=0$
- We can now provide an interpretation of the TVC

# Transversality Condition I

- The transversality condition is a complementary condition that must hold (in standard problems) in order for the consumption/savings plan of the individual agents to be optimal.
- In a finite-horizon problem,
  the TVC has a straightforward interpretation:
  the discounted value of the stock of capital left
  at the end of the planning period $(T)$ must be zero

$$a(T) \cdot e^{-\int_0^T (r(\nu)-n)\, d\nu} = 0$$

- As long as the interest rate is finite, the second term is positive, which reduces itself to the intuitive condition that $a_T = 0$.

# Transversality Condition II

- In the infinite horizon, we take the limit of
the finite-horizon condition as T tends to infinity:

$$\lim_{T \to \infty} \left[ a(T) \cdot e^{-\int_0^T (r(\nu) - n)\, d\nu} \right] = 0$$

- Interpretation: the PDV of assets at the "end of life" (infinity) must be zero. However, now $a(t)$ needs not converge to zero.
- A simple case in which the TVC holds is an economy converging to a steady-state where both $a(t)$ and $r(t)$ are constant
- However, the TVC can also hold if $a(t) \to \infty$
as long as the second term goes to zero "sufficiently fast"

# Equilibrium Prices I

- Equilibrium prices are given by

$$R(t) = f'(k(t)) \quad \text{and} \quad w(t) = f(k(t)) - k(t) f'(k(t)).$$

- Since $r(t) = R(t) - \delta$, then

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumer's EE, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta}$$

- Moreover, since $a(t) = k(t)$ and $\mu(t) = c(t)^{-\theta}$, the TVC can be written as

$$\begin{aligned} \lim_{t\to\infty} \left[\exp(-(\rho - n)t) \cdot \mu(t) \cdot a(t)\right] &= \\ \lim_{t\to\infty} \left[\exp(-(\rho - n)t) \cdot c(t)^{-\theta} \cdot k(t)\right] &= 0 \end{aligned}$$

# Equilibrium Prices II

- Finally, let us go back to the individual budget constraint

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t)$$

- And using the equilibrium conditions

$$\begin{aligned} a(t) &= k(t) \\ r(t) &= f'(k(t)) - \delta \\ w(t) &= f(k(t)) - k(t)\, f'(k(t)) \end{aligned}$$

- We conclude that

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t),$$

that can be interpreted as an aggregate resource constraint.

# Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\left(\rho - n\right)t\right) u\left(c\left(t\right)\right) dt,$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - (n+\delta)k\left(t\right) - c\left(t\right),$$

and $k\left(0\right) > 0$.

- Versions of the First and Second Welfare Theorems for economies with a continuum of commodities: solution to this problem should be the same as the equilibrium growth problem.
- Let us show the equivalence directly.

# Optimal Growth II

- Again set up the current-value Hamiltonian:

$$\tilde{H}(k, c, \mu) = u(c(t)) + \mu(t)\left[f(k(t)) - (n+\delta)k(t) - c(t)\right],$$

- The solution must satisfy

$$\begin{aligned} FOC_{PL} \quad &: \quad \tilde{H}_c(k, c, \mu) = 0 \\ &\Leftrightarrow \quad u'(c(t)) = \mu(t) \end{aligned}$$

and $EE_{PL}$:

$$\begin{aligned} \tilde{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) = \left(f'(k(t)) - \delta - n\right)\mu(t) \\ \Leftrightarrow \quad & \frac{\dot{\mu}(t)}{\mu(t)} = -\left(f'(k(t)) - \delta - \rho\right) \end{aligned}$$

$$RC: \quad \dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

$$TVC_{PL}: \quad \lim_{t\to\infty}\left[\exp\left(-(\rho - n)t\right) \cdot \mu(t) \cdot k(t)\right] = 0$$

# Optimal Growth III

- Assume CRRA. Repeating the same steps as before,

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta},$$

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t),$$

$$\lim_{t\to\infty} \left[ \exp(-(\rho - n)t) \cdot c(t)^{-\theta} \cdot k(t) \right] = 0$$

which are identical to the laissez-faire equilibrium conditions.

- Thus the competitive equilibrium is a Pareto optimum and the Pareto allocation can be decentralized as a competitive equilibrium.

# Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant. Thus:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k^*) - \delta - \rho}{\theta} = 0$$

$$\Longleftrightarrow f'(k^*) = \rho + \delta$$

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- Then, the resource constraint pins down the steady-state consumption level:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) = 0$$

$$\Longleftrightarrow c^* = f(k^*) - (n+\delta)k^*.$$

# Steady-State Equilibrium II

- A steady state where the capital-labor ratio and thus output are constant necessarily satisfies the TVC:

$$\lim_{t\to\infty}\left[\exp\left(-\left(\rho-n\right)t\right)\cdot k^{*}\cdot\left(c^{*}\right)^{-\theta}\right]=0$$

which is true as long as $\rho > n$.

# Transitional Dynamics I

- Equilibrium is determined by two differential equations:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta}$$

plus an initial condition, $k(0) > 0$, and a terminal condition:

$$\lim_{t\to\infty}\left[\exp(-(\rho - n)t) \cdot k(t) \cdot (c(t))^{-\theta}\right] = 0.$$

# Transitional Dynamics II

- Appropriate notion of *saddle-path stability*:
  - $c$ (or, equivalently, $\mu$) is the control variable, and $c(0)$ (or $\mu(0)$) is free: it has to adjust to satisfy transversality condition
  - If there were more than one path equilibrium would be indeterminate.
- Economic forces are such that indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.
- See Figure.

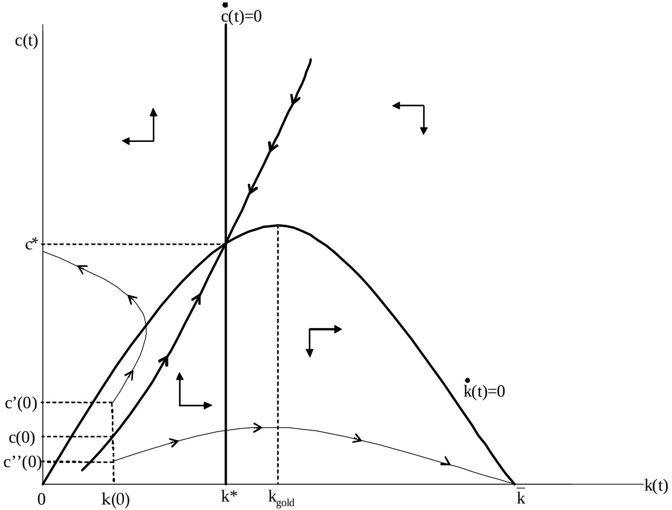


Figure: Transitional dynamics in the baseline neoclassical growth model

# Transitional Dynamics: Global Stability Analysis

- Intuitive argument:
  - if $c(0)$ started below it, say $c''(0)$, consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption). This would violate the transversality condition.
  - if $c(0)$ started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility (a little care is necessary with this argument, since necessary conditions do not apply at the boundary).

# Technological Change and the Canonical Neoclassical Model I

- Extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)],$$

where

$$A(t) = \exp(gt)A(0).$$

- Note: we assume labor-augmenting technological change. Else, there would be no balanced growth equilibrium

# Technological Change and the Canonical Neoclassical Model II

- Define $\hat{x}(t) \equiv X(t) / (A(t) L(t))$

$$\hat{y}(t) = F\left[\frac{K(t)}{A(t) L(t)}, 1\right] \equiv f\left(\hat{k}(t)\right),$$

- Assume CRRA preferences

# Equilibrium

- The equilibrium is now fully characterized by the following dynamic equations

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{1}{\theta}\left(f'\left(\hat{k}(t)\right) - \delta - \rho - \theta g\right),$$

$$\dot{\hat{k}}(t) = f\left(\hat{k}(t)\right) - (n + g + \delta)\,\hat{k}(t) - \hat{c}(t),$$

plus an initial condition, $\hat{k}(0) > 0$, and a terminal condition (TVC)

$$= \lim_{t\to\infty}\left\{\exp\left(-\left(\rho - n - (1-\theta)\,g\right)t\right)\cdot \hat{k}(t)\cdot \left(\hat{c}(t)\right)^{-\theta}\right\} = 0.$$

# Equilibrium (derivation EE, see book for more)

$$r(t) = f'\left(\hat{k}(t)\right) - \delta$$

- Since $c(t) = A(t) \cdot \hat{c}(t)$, then $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\hat{c}}(t)}{\hat{c}(t)} + g$
- Then:

$$\frac{\dot{\hat{c}}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho) \Longleftrightarrow \frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{1}{\theta}\left(f'\left(\hat{k}(t)\right) - \delta - \rho - \theta g\right)$$

# Steady-State

- In steady state, $f'\left(\hat{k}^*\right) = \rho + \delta + \theta g$.
- Pins down the steady-state value of the normalized capital ratio $\hat{k}^*$ uniquely.
- Normalized consumption level is then given by

$$\hat{c}^* = f\left(\hat{k}^*\right) - (n + g + \delta)\,\hat{k}^*,$$

- Per capita consumption grows at the rate $g$.
- The TVC now requires $\rho - n > (1 - \theta)\,g$.

# Technological Change and the Canonical Neoclassical Model XI

**Proposition** Consider the neoclassical growth model with labor augmenting technological progress at the rate $g$ and CRRA preferences. Suppose that $\rho - n > (1 - \theta) g$. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of $\hat{k}^*$, given by $f'\left(\hat{k}^*\right) = \rho + \delta + \theta g$, and output per capita and consumption per capita grow at the rate $g$.

# Transitional dynamics

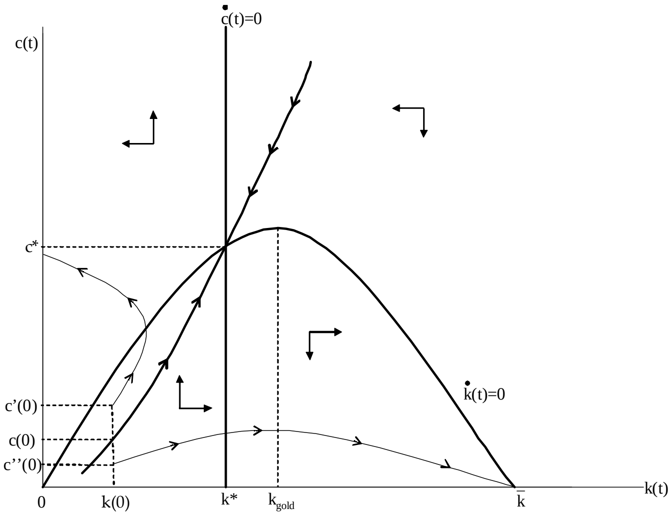


Figure: When g>0, simply replace $k$ and $c$ by $\hat{k}$ and $\hat{c}$

# Comparative Dynamics I

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- E.g.: Initial steady state represented by $(k^*, c^*)$. Unexpectedly, discount rate declines to $\rho' < \rho$.
- Following the decline $\hat{c}^*$ is above the stable arm of the new dynamic system: consumption must drop immediately

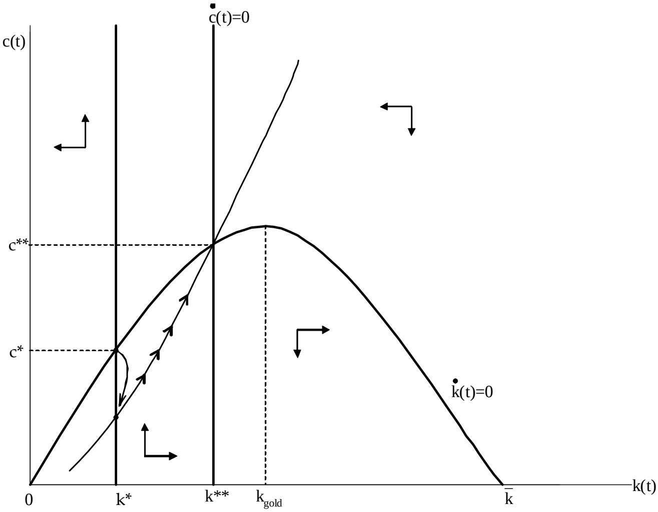


Figure: The dynamic response of capital and consumption to a decline in the discount rate from $\rho$ to $\rho' < \rho$.

# The Role of Policy I

- Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate $\tau$ and the proceeds of this are redistributed back to the consumers.
- Capital accumulation equation remains as above:

$$\dot{\hat{k}}(t) = f\left(\hat{k}(t)\right) - \hat{c}(t) - (n + g + \delta)\,\hat{k}(t),$$

- But interest rate faced by households changes to:

$$r(t) = (1 - \tau)\left(f'\left(\hat{k}(t)\right) - \delta\right),$$

# The Role of Policy II

- Growth rate of normalized consumption is then obtained from the consumer Euler equation

$$\begin{aligned}\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} &= \frac{1}{\theta}\left(r(t)-\rho-\theta g\right). \\ &= \frac{1}{\theta}\left((1-\tau)\left(f'\left(\hat{k}(t)\right)-\delta\right)-\rho-\theta g\right).\end{aligned}$$

- This implies

$$f'\left(\hat{k}^*\right) = \delta + \frac{\rho+\theta g}{1-\tau}.$$

- Since $f'(\cdot)$ is decreasing, higher $\tau$, reduces $\hat{k}^*$.
- Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.

# Appraisal neoclassical model

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- However, this model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.

# AK model I

- Neoclassical model: no autonomous engine of growth. In the absence of exogenous trend, growth dies off in the long-run.
  1. No theory of determinants of long-run growth;
  2. No theory of determinants of long-run cross-country differences in growth rates;
  3. Policies do not affect long-run growth.
- The AK-Model is a very simple model that can be viewed as the "limit case" of the neoclassical growth model. It provides the common analytical framework for a number of more interesting applications.

# AK model II

- Production technology (g=0):

$$f(k) = Ak$$

- Equilibrium interest rate is $r(t) = A - \delta$.
- Assume CRRA utility
- Given $k(0)$, a competitive equilibrium is determined by

$$\dot{c}(t) = \frac{A - \delta - \rho}{\theta} \cdot c(t) \qquad \text{(EE)}$$

$$\dot{k}(t) = Ak(t) - c(t) - (\delta + n)k(t) \qquad \text{(BC)}$$

$$\lim_{t \to \infty} \left[ \exp(-(\rho - n)t) \cdot k(t) \cdot (c(t))^{-\theta} \right] = 0 \qquad \text{(TVC)}$$

# AK model III

- We can obtain an explicit analytical solution:

  (a) Guess a steady-state solution such that $c/k$ is constant (assume $A > \delta + \rho$).

  $$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{A - \delta - \rho}{\theta} = \gamma$$

  (b) Use (BC)

  $$\frac{\dot{k}(t)}{k(t)} = (A - \delta - n) - \frac{c(t)}{k(t)}.$$

  (c) From (a)+(b):

  $$\frac{c(t)}{k(t)} = \frac{c}{k} = \rho - n - \frac{1-\theta}{\theta} \cdot [A - \delta - \rho]$$

  In particular:
  $c(0) = \left\{\rho - n - \frac{1-\theta}{\theta} \cdot [A - \delta - \rho]\right\} k(0)$.

# AK model IV

- Hence (a) and the solution for $c(0)$, we obtain analytical solutions:

$$c(t) = c(0) \cdot \exp\left(\left(\frac{A-\delta-\rho}{\theta}\right) t\right)$$

$$k(t) = k(0) \cdot \exp\left(\left(\frac{A-\delta-\rho}{\theta}\right) t\right)$$

- TVC (after replacing $k(t)$ by its solution):

$$\begin{aligned} 0 &= \lim_{t\to\infty} \left[\exp\left(-(\rho-n)t\right) \cdot \frac{k(t)}{c(t)} \cdot c(t) \cdot (c(t))^{-\theta} \cdot \right] \\ &= \lim_{t\to\infty} \left[\exp\left(-(\rho-n)t\right) \cdot \left(\frac{c}{k}\right)^{-1} \cdot (c(0))^{1-\theta} \right. \\ &\quad \left. \cdot \exp\left(\left(\frac{1-\theta}{\theta}(A-\delta-\rho)\right) t\right)\right] \end{aligned}$$

provided that the following condition (bounded utility) holds:

$$\rho > n + (1-\theta)(A-\delta-n)$$

- No transitional dynamics

# AK model V

- In this model, policies have permanent effects on growth.
- Consider again the introduction of a permanent tax on the returns on capital. The proceeds are rebated as lump-sums.
- The equilibrium interest rate is $r = (1 - \tau)(A - \delta)$, and the equilibrium growth rate is:

$$\gamma_\tau = \frac{(1 - \tau)(A - \delta) - \rho}{\theta}$$

# Two Simple AK Models

- Two simple models that deliver AK dynamics
- Assume n=g=0
- Basic human capital and knowledge spillovers

# Basic Human Capital Model I

- Suppose agents can accumulate both physical and human capital.
- Let the technology be

$$Y = F(K, H) = AK^{\alpha}H^{1-\alpha} = AK(H/K)^{1-\alpha}$$

# Basic Human Capital Model II

- Assume (unrealistically):
  1. Physical capital, human capital and consumption goods are produced with the same technology: One unit of final output can be used for consumption, investment in physical capital and investment in human capital.
  2. All investments are fully reversible.
  3. Same depreciation (rate $\delta$) for both types of capital (unimportant).

# Basic Human Capital Model III

- No arbitrage implies: $R_K = R_H = r + \delta$.
- Firms' profit-maximization:

$$R_K = \alpha A \left(H/K\right)^{1-\alpha} = \left(1-\alpha\right) A \left(H/K\right)^{-\alpha} = R_H$$

Solving for human-to-physical ratio yields:

$$H/K = \left(1-\alpha\right)/\alpha$$

- Hence, substituting away $H/K$:

$$\begin{aligned} r &= \alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha} A - \delta, \\ Y &= \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} AK \end{aligned}$$

# Basic Human Capital Model IV

- The equilibrium features

$$\dot{c}(t) = \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha} A - \delta - \rho}{\theta} \cdot c(t),$$
$$\dot{k}(t) = \left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} Ak(t) - c(t) - \delta k(t),$$

  plus a TVC

- In equilibrium, the economy grows at the constant rate

$$\gamma = \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha} A - \delta - \rho}{\theta}.$$

# Learning-by-doing Externalities I

- External effects of capital accumulation on productivity.
- As capital in firm $i$ accumulates, it has a productivity-enhancing effect on the capital installed in all firms.
- It becomes important to distinguish between firm-level and aggregate variables.
- Firm-level technology:

$$Y_i = F\left(K_i, \tilde{A}L_i\right)$$

- Labor-augmenting technical progress ($\tilde{A}$) is not firm-specific. We assume

$$\tilde{A} = \phi K$$

# Learning-by-doing Externalities II

- $\tilde{A}$ can be interpreted as public knowledge. Knowledge is assumed to have a non-rival character: when a firm adds to the stock of knowledge, all firms in the economy can benefit from this addition.
- Knowledge accumulation is assumed to be a pure spillover.
- For simplicity, we restrict attention to Cobb-Douglas technology:

$$F\left(K_i, \tilde{A}L_i\right) = K_i^{\alpha}\left(\tilde{A}L_i\right)^{1-\alpha} = AK_i^{\alpha}\left(KL_i\right)^{1-\alpha}$$

where $A \equiv \phi^{1-\alpha}$

# Learning-by-doing Externalities III

- Firms takes $\bar{K}$ as parametric. Thus, the equilibrium rates of return are:

$$R = F_{K_i}\left(K_i, \tilde{A}L_i\right) = \alpha A \left(\frac{KL_i}{K_i}\right)^{1-\alpha} \qquad \text{(LBD-FOC1)}$$

$$w = F_{L_i}\left(K_i, \tilde{A}L_i\right) = \frac{(1-\alpha) A K_i^{\alpha} \left(KL_i\right)^{1-\alpha}}{L_i} \qquad \text{(LBD-FOC2)}$$

- Assume a continuum of identical firms with total measure equal to $M$. Thus, in a symmetric equilibrium,

$$MK_i = K \text{ and } ML_i = L. \qquad \text{(LBD-EQ)}$$

# Learning-by-doing Externalities IV

- IMPORTANT: to characterize the competitive equilibrium, one must substitute (LBD-EQ) into (LBD-FOC1)-(LBD-FOC2) (i.e., atomistic firms ignore the effect of their investments on aggregate productivity).
- I.e., firms act in an uncoordinated fashion. So, using (LBD-EQ) to eliminate $K_i$ and $L_i$, leads to:

$$
\begin{aligned}
r &= R - \delta = \alpha A L^{1-\alpha} - \delta \\
w &= \frac{(1-\alpha) AK}{L^{\alpha}} = (1-\alpha) \cdot A \cdot k \cdot L^{1-\alpha}
\end{aligned}
$$

# Learning-by-doing Externalities V

- The equilibrium conditions are

$$\dot{c}(t) = \frac{\alpha A L^{1-\alpha} - \delta - \rho}{\theta} \cdot c(t)$$
$$\dot{k}(t) = \left(A L^{1-\alpha} - \delta\right) k(t) - c(t)$$

  plus a TVC

- The dynamics of this model are isomorphic to those of the AK model. But there are two differences:
  1. Scale effects
  2. Equilibrium is not Pareto optimal
     (discussed as an exercise).