Lecture 3: Growth with Overlapping Generations (*Acemoglu 2009*, Chapter 9, adapted from Zilibotti)

Kjetil Storesletten

September 5, 2014

Growth with Overlapping Generations

- In many situations, the assumption of a representative household is not appropriate.
- E.g., an economy in which new households arrive (or are born) over time.
- New economic interactions: decisions made by older "generations" will affect the prices faced by younger "generations".
- *Overlapping generations models*
  1. Capture potential interaction of different generations of individuals in the marketplace;
  2. Provide tractable alternative to infinite-horizon representative agent models;
  3. Some key implications different from neoclassical growth model;
  4. Dynamics in some special cases quite similar to Solow model rather than the neoclassical model;
  5. Generate new insights about the role of national debt and Social Security in the economy.

Problems of Infinity I

- Static economy with countably infinite number of households, $i \in \mathbb{N}$
- Countably infinite number of commodities, $j \in \mathbb{N}$.
- All households behave competitively.
- Household i has preferences:

$$u_i = c_i^i + c_{i+1}^i,$$

- c_j^i denotes the consumption of the jth type of commodity by household i.
- Endowment vector ω of the economy: each household has one unit endowment of the commodity with the same index as its index.
- Choose the price of the first commodity as the numeraire, i.e., $p_0 = 1$.

Problems of Infinity II

Proposition In the above-described economy, a price vector such that $\bar{p}_j = 1$ for all $j \in \mathbb{N}$ is a competitive equilibrium price vector and induces an equilibrium with no trade.

- **Proof:**
  - At the proposed price vector each household has an income equal to 1.
  - Therefore, the budget constraint of household i can be written as

$$c_i^i + c_{i+1}^i \leq 1.$$

  - This implies that consuming own endowment is optimal for each household,
  - Thus the unit price vector and no trade constitute a competitive equilibrium.

Problems of Infinity III

- However, this competitive equilibrium is not Pareto optimal. Consider an alternative allocation such that:
  - Household $i = 0$ consumes its own endowment and that of household 1.
  - All other households, indexed $i > 0$, consume the endowment of the neighboring household, $i + 1$.
  - All households with $i > 0$ are as well off as in the competitive equilibrium.
  - Individual $i = 0$ is strictly better-off.

Proposition In the above-described economy, the competitive equilibrium with no trade is not Pareto optimal.

Problems of Infinity IV

- A competitive equilibrium is not Pareto optimal...
  Violation of the First Welfare Theorem?
- The version of the FWT stated in the first lecture holds for a finite number of households
- Generalization to OLG economy requires an additional condition

Theorem **(First Welfare Theorem with** ∞ households and commodities**)** Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \omega, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ countably infinite. Assume that all households are locally non-satiated and that $\sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$. Then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is Pareto optimal.

- But in the proposed competitive equilibrium $p_j^* = 1$ for all $j \in \mathbb{N}$, so that $\sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i = \sum_{j=0}^{\infty} p_j^* = \infty$.

Problems of Infinity V

- The First Welfare Theorem fails in OLG economies due to the "problem of infinity".
- This abstract economy is “isomorphic” to the baseline overlapping generations model.
- The Pareto suboptimality in this economy will be the source of potential inefficiencies in overlapping generations model.

Problems of Infinity VI

- A reallocation of ω can achieve the Pareto-superior allocation as an equilibrium (second welfare theorem)
- Give the endowment of household $i \geq 1$ to household $i - 1$.
  - At the new endowment vector $\tilde{\omega}$, household $i = 0$ has one unit of good $j = 0$ and one unit of good $j = 1$.
  - Other households i have one unit of good $i + 1$.
- At the price vector $\bar{p}$, such that $p_j = 1 \; \forall j \in \mathbb{N}$, household 0 has a budget set

$$c_0^0 + c_1^1 \leq 2,$$

  thus chooses $c_0^0 = c_1^0 = 1$.
- All other households have budget sets given by

$$c_i^i + c_{i+1}^i \leq 1,$$

- Thus it is optimal for each household $i > 0$ to consume one unit of the good c_{i+1}^i
- Thus the allocation is a competitive equilibrium.

The Baseline Overlapping Generations Model

- Time is discrete and runs to infinity.
- Each individual lives for two periods.
- Individuals born at time t live for dates t and $t+1$.
- Assume a separable utility function for individuals born at date t,

$$U(t) = u(c_1(t)) + \beta u(c_2(t+1))$$

- $u(c)$ satisfies the usual Assumptions on utility.
- $c_1(t)$: consumption at t of the individual born at t when young.
- $c_2(t+1)$: consumption at $t+1$ of the same individual when old.
- β is the discount factor.

Demographics, Preferences and Technology I

- Exponential population growth,

$$L(t) = (1+n)^t L(0).$$

- For simplicity, let us assume Cobb-Douglas technology:

$$f(k(t)) = k(t)^{\alpha}$$

- Factor markets are competitive.
- Individuals only work in the first period and supply one unit of labor inelastically, earning $w(t)$.

Demographics, Preferences and Technology II

- Assume that $\delta = 1$.
- Then, the gross rate of return to saving, which equals the rental rate of capital, is

$$1 + r(t) = R(t) = f'(k(t)) = \alpha k(t)^{\alpha-1},$$

- As usual, the wage rate is

$$w(t) = f(k(t)) - k(t) f'(k(t)) = (1-\alpha) k(t)^{\alpha}.$$

Consumption Decisions I

- Assume CRRA utility. Savings is determined from

$$\max_{c_1(t), c_2(t+1), s(t)} U(t) = \frac{c_1(t)^{1-\theta} - 1}{1-\theta} + \beta \left(\frac{c_2(t+1)^{1-\theta} - 1}{1-\theta} \right)$$

subject to

$$c_1(t) + s(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)\, s(t),$$

- Old individuals rent their savings of time t as capital to firms at time $t+1$, and receive gross rate of return $R(t+1) = 1 + r(t+1)$
- Second constraint incorporates notion that individuals only spend money on their own end of life consumption (no altruism or bequest motive).

Consumption Decisions II

- Since preferences are non-satiated,
  both constraints will hold as equalities.
- Thus first-order condition for a maximum can be written in the familiar form of the consumption Euler equation,

$$\frac{c_2(t+1)}{c_1(t)} = (\beta R(t+1))^{1/\theta},$$

- or alternatively expressed in terms of saving function

$$\frac{R(t+1)s(t)}{w(t)-s(t)} = (\beta R(t+1))^{\frac{1}{\theta}}.$$

Consumption Decisions III

- Rearranging terms yields the following equation for the saving rate:

$$s(t) = s(w(t), R(t+1)) = \frac{w(t)}{\left[1 + \beta^{-1/\theta} R(t+1)^{-(1-\theta)/\theta}\right]},$$

- Note: $s(t)$ is strictly increasing in $w(t)$ and may be increasing or decreasing in $R(t+1)$.
- In particular, $s_R > 0$ if $\theta < 1$, $s_R < 0$ if $\theta > 1$, and $s_R = 0$ if $\theta = 1$.
- Reflects counteracting influences of income and substitution effects.

Consumption Decisions IV

- Total savings in the economy will be equal to

$$S(t) = K(t+1) = s(w(t), R(t+1)) L(t).$$

- $L(t)$ denotes the size of generation t, who are saving for time $t+1$.
- Since capital depreciates fully after use and all new savings are invested in capital.

Equilibrium Dynamics

- Recall that $K(t+1) = k(t+1) \cdot L(t) \cdot (1+n)$. Then,

$$\begin{aligned} k(t+1) &= \frac{s(w(t), R(t+1))}{(1+n)} \\ &= \frac{(1-\alpha)\, k(t)^{\alpha}}{(1+n)\left[1+\beta^{-1/\theta} k(t+1)^{(1-\alpha)(1-\theta)/\theta}\right]} \end{aligned}$$

- The steady state solves the following implicit equation:

$$k^* = \frac{(1-\alpha)(k^*)^{\alpha}}{(1+n)\left[1+\beta^{-1/\theta}(k^*)^{(1-\alpha)(1-\theta)/\theta}\right]}.$$

- In general, multiple steady states are possible.
  Multiplicity is ruled out assuming that $\theta \geq 1$.

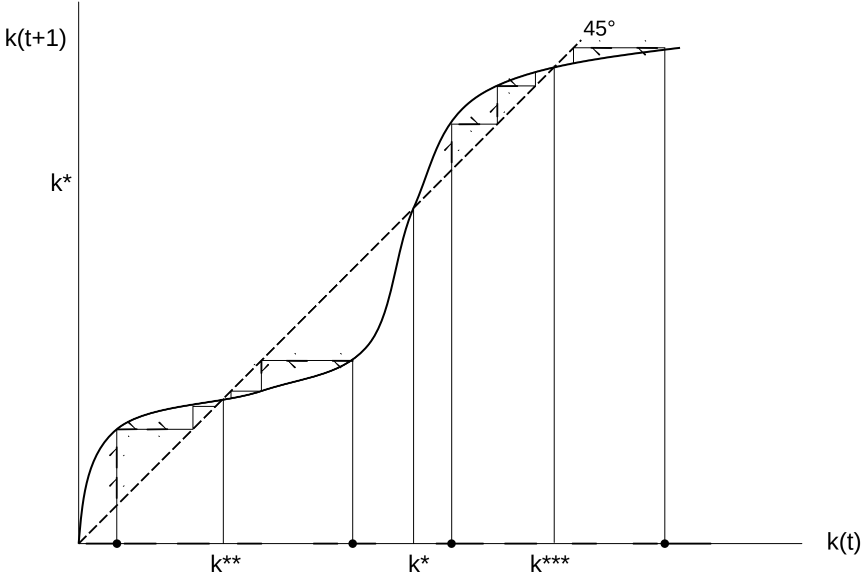


Figure: Multiple steady states in OLG models.

The Canonical Overlapping Generations Model I

- Many of the applications use log preferences ($\theta = 1$)

$$U(t) = \log c_1(t) + \beta \log c_2(t+1).$$

- Consumption Euler equation:

$$\frac{c_2(t+1)}{c_1(t)} = \beta R(t+1)$$

- Savings should satisfy the equation

$$s(w(t), R(t+1)) = \frac{\beta}{1+\beta} w(t),$$

- Constant saving rate, equal to $\beta/(1+\beta)$,
out of labor income for each individual.

The Canonical Overlapping Generations Model II

- The equilibrium law of motion of capital is

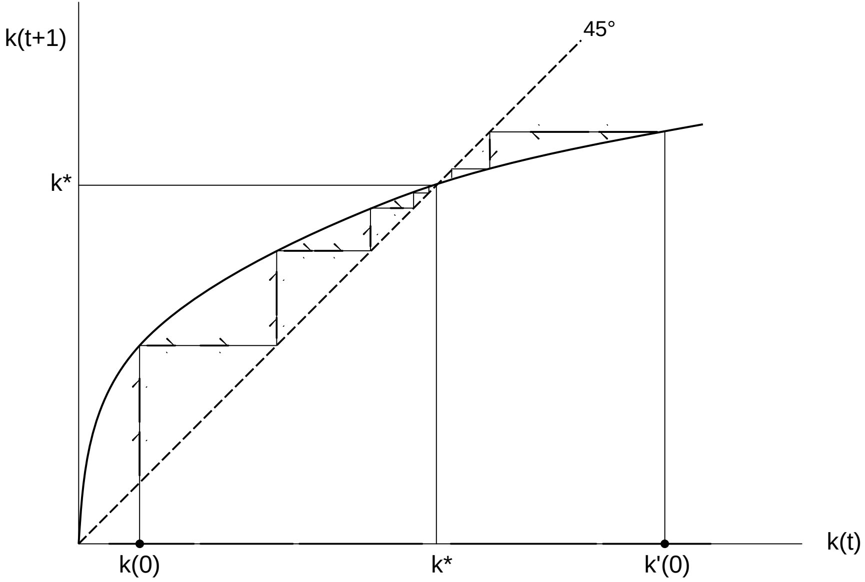
$$k(t+1) = \frac{\beta(1-\alpha)[k(t)]^{\alpha}}{(1+n)(1+\beta)}$$

- There exists a unique steady state with

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}.$$

The Canonical Overlapping Generations Model III

- Equilibrium dynamics are identical to those of the basic Solow model and monotonically converge to k^*.
- Income and substitution effects exactly cancel each other: changes in the interest rate (and thus in the capital-labor ratio of the economy) have no effect on the saving rate.
- Structure of the equilibrium is essentially identical to the Solow model.


k(t+1)
45°
k*
k(t)
k(0)
k*
k'(0)

Overaccumulation I

- Compare the overlapping-generations equilibrium to the choice of a social planner wishing to maximize a weighted average of all generations' utilities.
- Suppose that the social planner maximizes

$$\sum_{t=0}^{\infty} \beta_S^t \cdot L(t) \cdot (u(c_1(t)) + \beta u(c_2(t+1)))$$

subject to the resource constraint (Y=I+C)

$$F(K(t), L(t)) = K(t+1) + L(t)\, c_1(t) + L(t-1)\, c_2(t).$$

which can be rewritten as

$$f(k(t)) = (1+n)\, k(t+1) + c_1(t) + \frac{c_2(t)}{1+n}$$

- β_S is the discount factor of the social planner, which reflects how she values the utilities of different generations.

$$\sum_{t=0}^{\infty} \beta_S^t \left(u\left(c_1\left(t\right)\right) + \beta u\left(c_2\left(t+1\right)\right)\right)$$
$$= \quad ... + \beta_S^t \left(u\left(c_1\left(t\right)\right) + \beta u\left(c_2\left(t+1\right)\right)\right) + \beta_S^{t+1} \cdot ...$$

- Substituting away $c_1(t)$ and $c_2(t+1)$ using the constraint yields

$$... + \beta_S^t (1+n)^t \left(u\left(f\left(k\left(t\right)\right) - (1+n)\, k\left(t+1\right) - \frac{c_2\left(t\right)}{1+n}\right)\right.$$
$$+ \beta u\left(\left(1+n\right) f\left(k\left(t+1\right)\right) - \left(1+n\right)^2 k\left(t+2\right) - \left(1+n\right) c_1\left(t+1\right.\right.$$
$$+ \beta_S^{t+1} (1+n)^{t+1} \cdot ....$$

- The FOC w.r.t. $k(t+1)$ yields

$$u'\left(c_1\left(t\right)\right) = \beta f'\left(k\left(t+1\right)\right) u'\left(c_2\left(t+1\right)\right).$$

Overaccumulation II

- Social planner's maximization problem implies the following FOCs:

$$u'\left(c_1\left(t\right)\right) = \beta f'\left(k\left(t+1\right)\right) u'\left(c_2\left(t+1\right)\right).$$

- Since $R\left(t+1\right) = f'\left(k\left(t+1\right)\right)$, this is identical to the Euler Equation in the LF equilibrium.
- Not surprising: the planner allocates consumption of a given individual in exactly the same way as the individual himself would do.
- However, the allocations *across* generations will differ. Social planner's first-order conditions for allocation across generations:

$$\begin{aligned} u'\left(c_1\left(t\right)\right) &= \beta_S\left(1+n\right) f'\left(k\left(t+1\right)\right)\frac{u'\left(c_1\left(t+1\right)\right)}{1+n} \\ &\Rightarrow \\ \frac{u'\left(c_1\left(t\right)\right)}{u'\left(c_1\left(t+1\right)\right)} &= \beta_S \cdot f'\left(k\left(t+1\right)\right) \end{aligned}$$

Overaccumulation III

- Socially planned economy will converge to a steady state with capital-labor ratio k^S such that

$$f'\left(k^S\right) = \frac{1}{\beta_S},$$

- Identical to the Ramsey growth model in discrete time (if we reinterpret β_S , of course).
- k^S chosen by the planner does not depend on preferences nor on β.
- k^S will typically differ from equilibrium k^*.
- Competitive equilibrium is not in general Pareto optimal.

Overaccumulation IV

- Define k_{gold} as the steady state level of k that maximizes consumption per worker. More specifically, note that in steady state, the economy-wide resource constraint implies:

$$f\left(k^*\right) - (1+n)k^* = c_1^* + (1+n)^{-1} c_2^* \equiv c^*,$$

- Therefore

$$\frac{\partial c^*}{\partial k^*} = f'\left(k^*\right) - (1+n)$$

- k_{gold} is formally defined as

$$f'\left(k_{gold}\right) = 1+n.$$

- When $k^* > k_{gold}$, then $\partial c^* / \partial k^* < 0$: reducing savings can increase consumption for all generations.
- k^* can be greater than k_{gold}. Instead, $k^S < k_{gold}$.

Overaccumulation V

- If $k^* > k_{gold}$, the economy is said to be *dynamically inefficient*—it overaccumulates.
- Identically, dynamic inefficiency arises iff

$$r^* < n,$$

- Recall in infinite-horizon Ramsey economy, transversality condition required that $r > g + n$.
- Dynamic inefficiency arises because of the heterogeneity inherent in the overlapping generations model.
- Suppose we start from steady state at time T with $k^* > k_{gold}$.

Overaccumulation VI

- Consider the following variation: change next period's capital stock by $-\Delta k$, where $\Delta k > 0$, and from then on, we immediately move to a new steady state (clearly feasible).
- This implies the following changes in consumption levels:

$$\begin{aligned} \Delta c(T) &= (1+n)\Delta k > 0 \\ \Delta c(t) &= -\left(f'(k^* - \Delta k) - (1+n)\right)\Delta k \text{ for all } t > T \end{aligned}$$

- The first expression reflects the direct increase in consumption due to the decrease in savings.
- In addition, since $k^* > k_{gold}$, for small enough Δk, $f'(k^* - \Delta k) - (1+n) < 0$, thus $\Delta c(t) > 0$ for all $t \geq T$.
- The increase in consumption for each generation can be allocated equally during the two periods of their lives, thus necessarily increasing the utility of all generations.

Overaccumulation VII

Proposition In the baseline overlapping-generations economy, the competitive equilibrium is not necessarily Pareto optimal. More specifically, whenever $r^* < n$ and the economy is dynamically inefficient, it is possible to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.

- Pareto inefficiency of the competitive equilibrium is intimately linked with *dynamic inefficiency.*

Overaccumulation VIII

- Intuition for dynamic inefficiency:
  - Dynamic inefficiency arises from overaccumulation.
  - Results from current young generation needs to save for old age.
  - However, the more they save, the lower is the rate of return.
  - Effect on future rate of return to capital is a pecuniary externality on next generation
  - If alternative ways of providing consumption to individuals in old age were introduced, overaccumulation could be ameliorated.

Role of Social Security in Capital Accumulation

- Social Security as a way of dealing with overaccumulation
- Fully-funded system: young make contributions to the Social Security system and their contributions are paid back to them in their old age.
- Unfunded system or a *pay-as-you-go:* transfers from the young directly go to the current old.
- Pay-as-you-go (unfunded) Social Security discourages aggregate savings.
- With dynamic inefficiency, discouraging savings may lead to a Pareto improvement.

Fully Funded Social Security I

- Government at date t raises some amount $d(t)$ from the young, funds are invested in capital stock, and pays workers when old $R(t+1)d(t)$.
- Thus individual maximization problem is,

$$\max_{c_1(t),c_2(t+1),s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)(s(t) + d(t)),$$

for a given choice of $d(t)$ by the government.

- Notice that now the total amount invested in capital accumulation is $s(t) + d(t) = (1+n)k(t+1)$.

Fully Funded Social Security II

- No longer the case that individuals will always choose $s(t) > 0$.
- As long as $s(t)$ is free, whatever $\{d(t)\}_{t=0}^{\infty}$, the competitive equilibrium applies.
- When $s(t) \geq 0$ is imposed as a constraint, competitive equilibrium applies if given $\{d(t)\}_{t=0}^{\infty}$, privately-optimal $\{s(t)\}_{t=0}^{\infty}$ is such that $s(t) > 0$ for all t.
- A funded Social Security can increase – but not decrease – savings. It cannot lead to Pareto improvements.

Unfunded Social Security I

- Government collects $d(t)$ from the young at time t and distributes to the current old with per capita transfer $b(t) = (1+n)\,d(t)$
- Individual maximization problem becomes

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)\,s(t) + (1+n)\,d(t+1),$$

for a given feasible sequence of Social Security payment levels $\{d(t)\}_{t=0}^{\infty}$.

- Rate of return on Social Security payments is n rather than $r(t+1)$, because unfunded Social Security is a pure transfer system. If $r^* < n$ this is welfare improving.

Unfunded Social Security II

- Unfunded Social Security reduces capital accumulation.
- Discouraging capital accumulation can have negative consequences for growth and welfare.
- In fact, empirical evidence suggests that there are many societies in which the level of capital accumulation is suboptimally low.
- But here reducing aggregate savings may be good when the economy exhibits dynamic inefficiency.

Unfunded Social Security III

Proposition Consider the above-described overlapping generations economy and suppose that the decentralized competitive equilibrium is dynamically inefficient. Then there exists a feasible sequence of unfunded Social Security payments $\{d(t)\}_{t=0}^{\infty}$ which will lead to a competitive equilibrium starting from any date t that Pareto dominates the competitive equilibrium without Social Security.

- Similar to way in which the Pareto optimal allocation was decentralized in the example economy above.
- Social Security is transferring resources from future generations to initial old generation.
- But with *no* dynamic inefficiency, any transfer of resources (and any unfunded Social Security program) would make some future generation worse off.

Overlapping Generations with a Long-lived Asset

- Suppose there exists A units of a long-lived asset in the OLG economy ("land"). The asset pays a (constant) dividend $d(t) = d$ every period.
- Let $p^{e,i}(t+1)$ be the expectation of houshold i about the price per unit of the asset next period
  - Claim: all households will have the same expectations (assuming there are no frictions and no limits to betting),

$$p^{e,i}(t+1) = p^{e}(t+1)$$

  - Proof: if people held different expectations, they would bet against each other so as to align the expectations

Temporary equilibrium

- Consider the payoff from purchasing the asset today and selling it tomorrow, after collecting the dividend.
  - Cost of investment is $p(t)$
  - The (discounted) expected return on the investment is

$$\frac{p^e(t+1)+d}{R(t+1)}$$

  - Any equilibrium must have the expected return on the asset equal to the rate of return on private lending/bonds (otherwise there would be an arbitrage opportunity: borrow in the low-return asset and invest in the high-return asset):

$$R(t+1)=\frac{p^e(t+1)+d}{P(t)}$$

  - This gives us a new *equilibrium condition* for the price of the asset

Perfect foresight

- - *Definition 1*: a *temporary equilibrium* is a competitive equilibrium in period t, given an expected price $p^e(t+1)$ tomorrow.
  - *Definition 2*: A *perfect foresight competitive equilibrium* with land is an infinite sequence of prices $p(t)$, $R(t)$, and $w(t)$ and endogenous variables such that the time t values are a temporary equilibrium satisfying

$$p(t+1) = p^e(t+1)$$

Budget constraints

- Assume (for simplicity)
  - zero population growth
  - no government debt, taxes, or transfers
  - a pure endowment economy (no capital) where endowment when young is ω
  - The asset is initially held by the old (who sell it to the young).
- The individual budget constraints are then given by

$$\begin{aligned} c_1(t) &= \omega - p(t) \cdot a(t+1) \\ c_2(t+1) &= (p(t+1) + d) \cdot a(t+1), \end{aligned}$$

where $a(t+1)$ is the amount of the asset purchased by the young in period t.

Equilibrium conditions

1. 1. Aggregate savings equals aggregate supply of assets:

   $$S(t) = p(t) A$$

   and $a(t+1) = A$
2. The interest rate is given by the Euler equation,

   $$\frac{u'(c_1(t))}{u'(c_2(t+1))} = \beta R(t+1)$$

3. The price sequence satisfies

   $$p(t) = \frac{p(t+1) + d}{R(t+1)}$$

Finding an equilibrium

- Guess a price p_t and verify that the equilibrium conditions are satisfied for the p_{t+1}, p_{t+2}, ... implied by the equilibrium condition, expressed as a combination of the equilibrium conditions, $p(t) = f(p(t+1), d, A)$. In our example,

$$\frac{u'(c_1(t))}{u'(c_2(t+1))} = \beta R(t+1) = \beta \frac{p(t+1) + d}{P(t)}$$

- The economy impose some natural restrictions on the price sequence, such as ruling out negative prices or price sequences that are explosive: there typically exists some upper bound on how large prices can be (somebody must be able to pay the price).

A parametric example

Assume $u(c) = c - \frac{b}{2}c^2$, implying

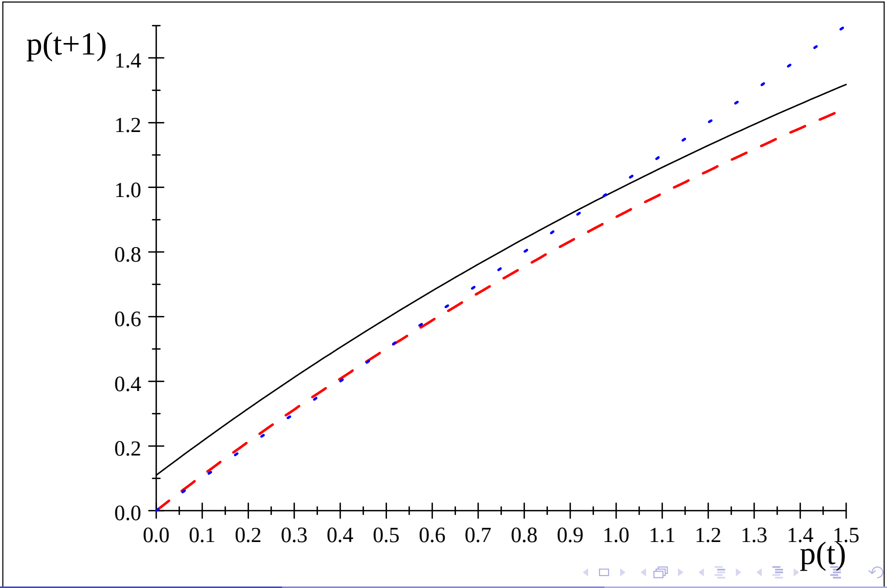
$$\frac{1 - bc_1(t)}{1 - bc_2(t+1)} = \frac{1 - b\left[w - p(t)A\right]}{1 - bp(t+1)A} = \beta \frac{p(t+1) + d}{p(t)} \Rightarrow$$

Solving yields

$$p(t) = \frac{-(1 - bw) + \sqrt{(1 - bw)^2 + 4bA\beta(p(t+1) + d)(1 - bp(t+1)A}}{2bA}$$

Graphical illustration of equilibrium

Numerical example $d = 0.1$ (black solid) and $d = 0$ (red dashed):



Stationary equilibria

- Simplification: restrict attention to *stationary equilibrium*:
- Suppose there is a stationary equilibrium with a constant interest rate R and a constant asset price p. The price-sequence condition $p(t) = [p(t+1) + d] / R(t+1)$ then becomes

$$p = \frac{p+d}{R} \tag{1}$$

- To clear the market for the asset, the young must buy all of it (there are no other potential buyers). The consumption allocation then becomes

$$\begin{aligned} c_1(t) &= \omega - pA = c_1 \\ c_2(t+1) &= (p+d)A = c_2, \end{aligned}$$

This allocation implies the following (equilibrium) interest rate:

$$\frac{u'(\omega - pA)}{u'((p+d)A)} = \beta R = \beta \frac{p+d}{p} \tag{2}$$

Two cases

1. The asset (land) yields some dividends, $d > 0$, and the interest rate is positive ($R > 1$). Then equation (1) becomes
$$p = \frac{d}{R-1},$$
i.e., the price is the present value of the future dividends.
   - Note: when $d > 0$, the interest rate cannot be zero since this would imply that land becomes infinitely expensive ($p \to \infty$). Since p cannot be negative, $R < 1$ is ruled out, too.
2. Land does not yield any dividends ($d = 0$). Then equation (1) becomes
$$p = \frac{p}{R}.$$

Two possible stationary equil. when d=0

- 1. 1. Autarky: $p = 0$. Agents eath their endowments when young and old (regardless of R and the endowments)
  2. Bubble: $R = 1$. This implies an Euler equation (2) of

$$\frac{u'(\omega - pA)}{u'(pA)} = \beta,$$

Simplify by setting $\beta = 1$ which implies $\omega - pA = pA$ and

$$p = \frac{\omega}{2A}.$$

I.e., equal consumption across generations: $c_1 = c_2 = \omega/2$. Note: the asset has a positive price even if it will never pay a dividend. This is a *rational bubble*.

Lessons

1. Rational bubbles can arise only if the interest rate is sufficiently low (lower than the growth rate of the economy)
2. Bubbles are good: it is an alternative to government debt and pay-as-you-go pensions to deal with dynamic inefficiency.
3. Bubbles can burst (if people suddenly starts believing in $p = 0$, then the game is over) and this gives a welfare loss

Overlapping Generations with Perpetual Youth I

- In baseline overlapping generation model individuals have finite lives and know when will die.
- Alternative model along the lines of the "Poisson death model" or the *perpetual youth model.*
- Discrete time.
- Each individual faces a probability $\nu \in (0,1)$ that his life will come to an end at every date (these probabilities are independent).
- Expected utility of an individual with a "pure" discount factor β is given by

$$\sum_{t=0}^{\infty} (\beta (1-\nu))^{t} u(c(t)).$$

Overlapping Generations with Perpetual Youth II

- Since the probability of death is ν and is independent across periods, the expected lifetime of an individual is:

$$\text{Expected life} = \nu + 2(1-\nu)\nu + 3(1-\nu)^2\nu + ... = \frac{1}{\nu} < \infty.$$

- With probability ν individual will have a total life of length 1, with probability $(1-\nu)\nu$, he will have a life of length 2, and so on.
- Individual i's flow budget constraint,

$$a_i(t+1) = (1 + r(t+1))\, a_i(t) - c_i(t) + w(t) + z_i(t),$$

- $z_i(t)$ is a transfer to the individual which is introduced because individuals face an uncertain time of death, so there may be "accidental bequests".
- One possibility is accidental bequests are collected by the government and redistributed equally across all households in the economy.

Overlapping Generations with Perpetual Youth III

- But this would require a constraint $a_i(t) \geq 0$, to prevent accumulating debts by the time their life comes to an end.
- Alternative (Yaari and Blanchard):
  introducing life-insurance or annuity markets.
  - Company pays $z(a(t))$ to an individual during every period in which he survives.
  - When the individual dies, all his assets go to the insurance company.
  - $z(a(t))$ depends only on $a(t)$ and not on age from perpetual youth assumption.
- Profits of insurance company contracting with an individual with $a(t)$, at time t will be

$$\pi(a, t) = -(1-\nu) z(a) + \nu(1 + r(t+1)) a.$$

Overlapping Generations with Perpetual Youth IV

- With free entry, insurance companies should make zero expected profits, requires that $\pi(a(t), t) = 0$ for all t and a, thus

$$z(a(t)) = \frac{\nu}{1-\nu}(1 + r(t+1))\, a(t).$$

- Since each agent faces a probability of death equal to ν at every date, there is a natural force towards decreasing population.
- Assume new agents are born, not into a dynasty, but become separate households.
- When population is $L(t)$, assume there are $nL(t)$ new households born.
- Consequently,

$$L(t+1) = (1 + n - \nu)\, L(t),$$

with the boundary condition $L(0) = 1$.

- We assume that $n \geq \nu$ (non-declining population)

Overlapping Generations with Perpetual Youth V

- Perpetual youth and exponential population growth leads to simple pattern of demographics in this economy.
- At some point in time $t > 0$, there will be $n(1+n-\nu)^{t-1}$ one-year-olds, $n(1+n-\nu)^{t-2}(1-\nu)$ two-year-olds, $n(1+n-\nu)^{t-3}(1-\nu)^2$ three-year-olds, etc.
- Standard production function with capital depreciating at the rate δ. Competitive markets.
- As usual: $R(t) = f'(k(t))$, $r(t+1) = f'(k(t)) - \delta$, and $w(t) = f(k(t)) - k(t) f'(k(t))$.
- Allocation in this economy involves $\{K(t), w(t), R(t)\}_{t=0}^{\infty}$, but consumption is not the same for all individuals.

Overlapping Generations with Perpetual Youth VI

- Denote the consumption at date t of a household born at date $\tau \leq t$ by $c(t \mid \tau)$.
- Allocation must now specify the entire sequence $\{c(t \mid \tau)\}_{t=0,\tau \leq t}^{\infty}$.
- Using this notation and the life insurance contracts introduced above, the flow budget constraint of an individual of generation τ can be written as:

$$
\begin{aligned}
a(t+1 \mid \tau) &= (1+r(t+1))(1+r(t))\, a(t \mid \tau) + \frac{\nu}{1-\nu}(1+r(t - \\
&= \frac{1+r(t+1)}{1-\nu} \cdot a(t \mid \tau) - c(t \mid \tau) + w(t).
\end{aligned}
$$

Overlapping Generations with Perpetual Youth VII

- Gross rate of return on savings is $1 + r(t+1) + \nu / (1 - \nu)$ and effective discount factor is $\beta (1 - \nu)$, so Euler equation is

$$\frac{u'(c(t \mid \tau))}{u'(c(t+1 \mid \tau))} = \beta (1 - \nu) \frac{1 + r(t+1)}{1 - \nu} = \beta (1 + r(t+1)) .$$

- Differences: applies separately to each generation τ and term ν.
- Different generations will have different levels of assets and consumption.
- With CRRA utility, all agents have the same consumption growth rate.

Conclusions

- OLG models fall outside the scope of the First Welfare Theorem:
  - they were partly motivated by the possibility of Pareto suboptimal allocations.
- Equilibria may be "dynamically inefficient" and feature overaccumulation: unfunded Social Security can ameliorate the problem.
- Declining path of labor income important for overaccumulation, and what matters is not finite horizons but arrival of new individuals.
- Overaccumulation and Pareto suboptimality: pecuniary externalities created on individuals that are not yet in the marketplace.
- Not overemphasize dynamic inefficiency: major question of economic growth is why so many countries have so little capital.