Problem Set 1
Due 3. September, 2014 (start of the seminar)

Exercise 1.1: The Welfare Theorems

Consider an economy consisting of a finite number of N households each with preferences over consumption that can be represented by the utility function

$$U\left(\{c_t^i\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \beta^t \log\left(c_t^i\right), \tag{1}$$

with $\beta \in (0,1)$, where c_t^i denotes consumption of household i in period t. The economy starts with an endowment of Y units of the final good and has no access to production technologies. This endowment can be stored without depreciating or gaining interest rate between periods.

(a) Define the competitive equilibrium for this economy. What are the Arrow-Debreu (AD) commodities?

(b) Characterize the set of *all* Pareto optimal allocations for this economy (denote the Pareto weight for each agent by $\lambda^i \geq 0$ where $\sum_{i=1}^{N} \lambda^i = 1$).

(c) According to Acemoglu (2009, Theorem 5.7), does the Second Welfare Theorem apply to this economy? (hint: check whether the truncation assumption in (iii) is satisfied.)

(d) Consider now a particular distribution of endowments $\{\omega^i\}_{i=1}^N$ such that $\sum_{i=1}^N \omega^i = Y$. Given these endowments, calculate the unique competitive equilibrium price vector and the corresponding consumption allocations.

(e) Consider the following allocation of endowments

$$\omega^1 = (3/4)Y,\ \omega^2 = (1/4)Y,\ \omega^i = 0\ \forall i \geq 3.$$

Write down the consumption allocation. What Pareto weights would correspond to this consumption allocation?

(f) Prove that there is aggregation in this economy (use Gorman's aggregation theorem and find the appropriate monotone transformation of the indirect utility function), and prove the existence of a normative representative household. Verify that the aggregate allocation of consumption and the price sequence does not depend on the initial distribution of endowments, $\{\omega^i\}_{i=1}^N$.

(g) Are all competitive equilibria (CE) Pareto optimal (PO)?

(h) Derive a redistribution scheme for decentralizing the entire set of Pareto optimal allocations. That is, compute the initial distribution of wealth associated with each Pareto optimal allocation of consumption.

Exercise 1.2: Arrow-Debreu vs. Sequential Trading Equilibrium

Consider a dynamic exchange economy similar to the one outlined in class (Chapter 5, slides 45-49). There are two periods, $t \in \{0, 1\}$ with two goods, $j \in \{x, z\}$ at each date. Denote the consumption of good j by household i in period t as $c^i_{j,t}$. Goods are perishable, so they must be indeed consumed in period t. Each household has a vector of endowments, $y^i_{j,t}$, and there are only two types of households, $i \in \{A, B\}$, that differ in preferences and endowments. Preferences can be represented by the utility function

$$U^i = c^i_{x,0} + \alpha^i \log\left(c^i_{z,0}\right) + \beta^i \left(\log\left(c^i_{x,1}\right) + \alpha^i \log\left(c^i_{z,1}\right)\right).$$

(a) Solve for the Arrow-Debreu (AD) equilibrium, using the good of type x in period 0 as the Numéraire, $p_{x,0} = 1$.

(b) Compare and comment on the following three cases in the AD equilibrium, given equal endowments $y^A_{j,t} = y^B_{j,t}$, $\alpha^A + \alpha^B = \alpha$, $\beta^A + \beta^B = \beta$,

   (i) $\alpha^A = \alpha^B$, $\beta^A = \beta^B$

   (ii) $\alpha^A > \alpha^B$, $\beta^A = \beta^B$

   (iii) $\alpha^A = \alpha^B$, $\beta^A > \beta^B$.

(c) Solve for the sequential trading equilibrium, using $p_{x,t} = 1$ as the Numéraire in the two sequential periods. Denote by q the price (in terms of the Numéraire) and by b^i the quantity of the household's bond holdings purchase in period 0.

(d) Compare and comment on these two type of equilibria.

Exercise 1.3: Aggregation

Consider an endowment economy consisting of a continuum of households of two types, A and B, each with unit mass. Households have preferences over consumption that can be represented by the utility function

$$\sum_{t=0}^{\infty} \left(\beta^i\right)^t \log\left(c^i_t\right), \quad i \in \{A, B\},$$

where type A households are assumed to be weakly more patient than those of type B, $\beta^A \geq \beta^B$.

(a) Suppose that the aggregate endowment of the economy is $Y_t = 2$ in each period t and the good is perishable, i.e. cannot be transferred to later periods. Characterize the Pareto optimal allocations of this economy when the planner puts the same weight on the households of each type (but different ones on households of different types).

(b) Consider the case where $\beta^A > \beta^B$ and $\lambda^A = \lambda^B$. What happens to the relative consumption level of the less patient households as $t \to \infty$? What is the relative consumption level in period 0?

(c) Assume that households can trade one-period discount bonds denoted in terms of consumption. Derive the present value budget constraint of a household by substituting out the bond quantities from the period-by-period budget constraints.

(d) Suppose the initial bond holding are equal to zero, $b_0^i = 0$. What is the present value of the transfer of wealth across agents that is required to implement the above planner allocation as a sequential competitive equilibrium?

(e) Suppose that $y_t^i = 1$ in every period. Find the relative planner weight $\tilde{\lambda}^i$ that would imply a zero wealth transfer.

(f) Suppose that $\beta^A > \beta^B$ and $y_t^i = 1$. Is the real interest rate of this economy increasing or decreasing over time? What is the steady-state interest rate of this decentralized economy? What is the path of the interest rate if $y_t^A = 2$ and $y_t^B = 0$? Do you conclude that this economy features aggregation, i.e., is the sequence of interest rates independent of the income distribution?