Problem Set 6
Due 12. November, 10:15

Exercise 6.1: Complete Markets and the Representative Household

The purpose of this exercise is to show that when households have preferences of the constant relative risk aversion (CRRA) form and markets are complete, we can use a representative household to summarize the behavior of all households.[1] Consider an economic environment with complete markets. Assume that households indexed by $i \in I$ trade all state-contingent claims at time 0 such that a household's maximization problem is of the following form:

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) \frac{c_t^i(s^t)^{1-\gamma}}{1-\gamma},$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) \left[c_t^i(s^t) - y_t^i(s^t) \right] = 0 \quad \forall i \in I.$$

We refer to LS, Ch. 8 for the explanation of the notation. In general, you will also find the relevant discussion on complete markets and aggregation related to this exercise in LS, Ch 8.

(a) Show that the Arrow-Debreu prices $q_t^0(s^t)$ can be written as functions of aggregate consumption only. Proceed in the following steps: (i) derive the Euler equation of an individual household i, taking as given prices, $q_t^0(s^t)$, and (ii) substitute individual consumption for aggregate consumption (hint: exploit the equilibrium relationship between individual and aggregate consumption in the case of CRRA preferences).

(b) Assume the existence of a representative household. Are the following statements correct? (i) "If we know the wealth holdings of the average household, we also know the wealth holdings of all other households." (ii) "The wealth distribution does not matter for the aggregate decisions of the households sector."

Suppose from now that individuals also make a labor supply choice, $h_t^i(s^t)$ (hours per period), and face an individual-specific but fixed wage w^i. Thus, labor income each period is $y_t^i(s^t) = w^i h_t^i(s^t)$. Moreover, assume that the instantaneous preferences over consumption and leisure are given by

$$u(c, h) = \log(c) + \log(1-h),$$

such that the objective function of the household reads

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) \left[\log\left(c_t^i(s^t)\right) + \log\left(1 - h_t^i(s^t)\right) \right].$$

[1] Notice that in general, we do not need to assume CRRA preferences. Homotheticity in combination with complete markets is sufficient to guarantee the existence of a representative agent (for example, see LS, Ch. 8).

(c) Solve for the competitive equilibrium prices and allocations (hint: you don't need to, but you can exploit the planner formulation of the problem and choose the right planner weights).

(d) Compute aggregate consumption and aggregate labor supply, C and H, of this economy, and show that the dispersion of wages is irrelevant for the aggregate variables. Provide an intuition for why aggregation holds in this economy.

Exercise 6.2: Lucas tree asset pricing and the equity premium

Consider a model in which there are three periods denoted by $t = 0, 1, 2$, and a unit mass of identical agents. There is risk about the realization of the state $s_t \in \{s_G, s_B\}$ in period 1 and 2. Each state realization occurs with probability $\pi(s_t)$ independent of calendar time. Note that since we have only two states, $\pi(s_G) = 1 - \pi(s_B)$. Each agent is endowed with a tree at the beginning of period 0, which gives the right to collect its dividends $d_t(s_t)$, in periods $t = 1, 2$ if the corresponding state, s_t, realizes. Agents also receive a riskless endowment e_t in each period and have the same preferences over consumption

$$\begin{aligned} EU &= \frac{c_0^{1-\gamma}}{1-\gamma} + E\left\{\beta\frac{c_1(s^t)^{1-\gamma}}{1-\gamma} + \beta^2\frac{c_2(s^t)^{1-\gamma}}{1-\gamma}\right\}, \\ &= \frac{c_0^{1-\gamma}}{1-\gamma} + \sum_{t=1}^{2}\sum_{s^t\in S^t}\beta^t\pi(s^t)\frac{c_t(s^t)^{1-\gamma}}{1-\gamma}, \quad \pi(s^t) \equiv \Pi_{k=1}^{t}\pi(s_t), \end{aligned}$$

where $\gamma \geq 0$ denotes the relative risk aversion and $0 < \beta \leq 1$ the objective discount factor of agents. Markets are competitive and financial markets are complete. For each risky period, $t = 1, 2$, there exists an Arrow security, $a_t(s^t)$ for each history of states, $s^t = (s_0, ..., s_t) = (s^{t-1}, s_t)$. The state-by-state budget constraints of the agents read

$$\begin{aligned} c_0 + \sum_{t=1}^{2}\sum_{s^t\in S^t} q_t^0(s^t)a_t(s^t) &= e_0 \\ c_1(s^1) &= e_1 + a_1(s^1) + d_1(s_1),\ \forall s^1 \in S^1, \\ c_2(s^2) &= e_2 + a_2(s^2) + d_2(s_2),\ \forall s^2 \in S^2. \end{aligned}$$

Note that in period 1 there are two possible realizations of history, $s^1 = (s_0, s_1)$, and in period 2 there are four possible realizations of the history, $s^2 = (s^1, s_2)$, as $s_t \in \{s_G, s_B\}$ for $t = 1, 2$.

(a) Compute the equilibrium level of consumption $c_t(s^t)$ for every possible history of realized states, s^t, in every period t, as well as the price of the Arrow securities, $q_t^0(s^t)$, in periods $t = 1, 2$. (Hint: you can base your answer on the fact that the period-by-period budget constraints can be summarized as

$$c_0 - e_0 + \sum_{t=1}^{2}\sum_{s^t\in S^t} q_t^0(s^t)\left[c_t(s^t) - (e_t + d_t(s_t))\right] = 0,$$

a result that follows from the complete markets assumption.)

(b) Compute the price of the tree in period 0, $p_0 \equiv p_0(s_0)$, and the prices of the tree in period 1, $p_1(s^1)$. (Hint: the dividend payments of the tree are equivalent to buying a linear combination of Arrow securities for each state. So, you can use the prices of the Arrow securities to price the tree.)

(c) Compute the price of a one period risk-free discount bond in period 0, $p_0^f(s_0)$, and the one-period discount bond prices $p_1^f(s^1)$ in period 1. (Hint: same here, use the Arrow security prices to price the discount bond.)

Assume from now the following specification of endowments and dividends: $(e_0, e_1, e_2) = (1, 0, 1)$ and $d_1(s_1) = 1$, $d_2(s_G) = 2$, $d_2(s_B) = 0$. Furthermore suppose that the probability of each state is symmetric, $\pi(s_G) = \pi(s_B) = 1/2$, and that there is not discounting of time, $\beta = 1$.

(d) Show that there is no equity premium in period 0,

$$R_0^f \equiv 1/p_0^f = \frac{\frac{1}{2}\left[p_1(s_0, s_G) + d_1(s_G)\right] + \frac{1}{2}\left[p_1(s_0, s_B) + d_1(s_B)\right]}{p_0} \equiv R_0.$$

(the expected return on the risk-free bond, R_0^f and the tree, R_0, are the same) and explain why. Can a different combinations of period 2 primitives (utility function, dividend, endowment) generate a positive premium?

(e) Compute the value of γ that generates an equity premium of

$$\frac{R_1(s^0, s_G) - R_1^f(s^0, s_G)}{R_1^f(s^0, s_G) - 1} = 1.5,$$

which corresponds to a risk-premium of 150%, in period 1 of the good state (this is the history (s_0, s_G)).

(f) What happens to this equity premium if $e_2 \to 0$. How does the value of γ (compared to the one in part (e)) change in this case? What happens to the equity premium if $e_2 \to \infty$ instead. Explain why.

(g) Which value of γ equates the one-period risk-free return in period 0, R_0^f, to the one in the good state of period 1, $R_1^f(s^0, s_G)$?

(h) Compute the return of a two-period risk-free discount bond in period 0. For which values of γ is the term structure flat/upward sloping/downward sloping, and explain why. Is there a risk-premium for the two-period discount bond?