# Problem Set 7 (Solution)
## Due 26. November, 10:15

### Exercise 7.1: An Aiyagari model with endogenous labor supply

Let us consider a closed economy with ex-ante identical households that - conditional on the states $(a_t, y_t)$ - solve the same recursive consumer problem in every period $t$

$$V(a_t, y_t) = u(c_t) - v(h_t) + \beta \mathrm{E}_t \left[ V(a_{t+1}, y_{t+1}) \right],$$

subject to

$$\begin{aligned} c_t + a_{t+1} &= (1+r)a_t + h_t y_t w \\ y_{t+1} &\sim \Gamma(y_t), \quad y_{t+1} \in Y, \\ c_t &\geq 0,\, h \geq 0,\, a_{t+1} \geq -b, \end{aligned}$$

where $c_t$ is individual consumption, $a_t$ asset holdings, $h_t$ labor supply, and $y_t$ can be interpreted as an idiosyncratic productivity shock that follows a stochastic Markov process, $\Gamma(y_t)$, with realizations drawn from the finite-valued set $Y$. The parameters are restricted to be, $0 < \beta < 1$ (subjective discount factor) and $b \geq 0$ (borrowing constraint), and the prices satisfy $w > 0$ and $0 < 1 + r < 1/\beta$. What makes households ex-post heterogeneous is that each of them draws an individual-specific productivity realization $y_{t+1}$ from the same distribution, and since financial markets are incomplete (there is only a risk-free asset available, but no state-contingent assets) agents with different shock histories will be different ex post.

(a) Derive the household's optimality conditions with respect to consumption, labor supply, and future asset holdings. (Hint: you can ignore the positivity constraints on consumption and labor supply, the functional forms of utility will make sure later this is satisfied. But you should incorporate the borrowing constraint on the asset holdings.)

(b) Let the inverse function of the marginal disutility of labor be denoted by $(v')^{-1}(\cdot)$. Characterize the optimal labor supply, as a function of the optimal consumption and the realization of the productivity shock, $\mathcal{H}(c_t, y_t)$.

(c) Guess a decision rule (superscripts indicate functions that depend on this guess) for the optimal consumption as a function of the future states,

$$c^0(a_{t+1}, y_{t+1}).$$

Let the future asset level be consistent with the borrowing constraint, $a_{t+1} \geq -b$ and the shock realizations be drawn from the finite valued set, $Y = \{\bar{y}_1, ..., \bar{y}_N\}$ with transition probabilities

$$\{\pi(y_{t+1} = \bar{y}_m | y_t = \bar{y}_n)\}_{n,m \in \{1,...,N\}}.$$

Denote the inverse marginal utility function of consumption by $(u')^{-1}(\cdot)$. Derive the function, $\mathcal{C}^0(a_{t+1}, y_t)$, that measures the current consumption level as a function of the future state, $a_{t+1}$, the current state, $y_t$, and the guess.

(d) Now find the current asset level $\mathcal{A}^0(a_{t+1}, y_t)$ that is consistent with the guess of the consumption decision rule. Or, on other words, $\mathcal{A}^0(a_{t+1}, y_t)$ defines the current assets holdings for a household with labor productivity, $y_t$, who has chosen to save $a_{t+1}$ for tomorrow).

(e) Use all the information above to update your guess on the consumption decision rule (characterize the current consumption level as a function of the current asset level, and the current shock realization, $c^1\left(\mathcal{A}^0(a_{t+1}, y_t), y_t\right)$). Be careful with future asset levels $a_{t+1}$ that are smaller than the lowest current asset levels that is exactly consistent with a binding borrowing constraint in the future

$$a_{t+1} \leq \mathcal{A}^0(-b, y_t).$$

For those asset levels, update the consumption function that is consistent with a binding borrowing constraint, $a_{t+1} = -b$.

(f) From here we aim to solve this simple incomplete markets economy with Matlab. Consider the following explicit form for the marginal utility functions

$$u'(c) = c^{-\gamma}, \quad \gamma = 3/2$$
$$v'(h) = h^{1/\varphi}, \quad \varphi = 2/3.$$

Set $\beta = 97/100$, $r = (1/\beta - 1) - 10^{-4}$, $w = 1$, $Y \equiv \{\bar{y}_1, \bar{y}_2\} = \{95/100, 105/100\}$, and let the state transition probabilities be symmetric,

$$\rho \equiv \pi(\bar{y}_1|\bar{y}_1) = \pi(\bar{y}_2|\bar{y}_2) = 9/10.$$

For the discretization of the future asset level consider $M = 250$ equally spaced grid points on the grid

$$A = \{\bar{a}_1, \ldots, \bar{a}_M\}, \quad \bar{a}_1 < \cdots < \bar{a}_M,\ \bar{a}_1 = -b,\ \bar{a}_M = 25,$$

where $b = 0$. Start with the guess that agents consume their asset income,

$$c^0(a_{t+1}, y_{t+1}) = ra_{t+1} + y_{t+1}.$$

Assuming that $a_{t+1}$ and $y_{t+1}$ can only take values on the corresponding grids (that's what is usually meant by discretization), $A$ and $Y$, respectively, this can be expressed as

$$c^0(A, Y) = rA + Y.$$

Use the results from the previous subquestions to write a program that updates this consumption function iteratively until the following convergence criterion is satisfied,

$$\left\| \frac{c^k(A, Y) - c^{k-1}(A, Y)}{c^{k-1}(A, Y)} \right\|_2 \leq 10^{-6},$$

where $k$ denotes the current iteration. Note that the endogenous values for the current asset level, $\mathcal{A}^k(A, Y)$, will in general be off the grid points of $A$, thus you will have to use an interpolation routine to update

$$c^k(a_{t+1}, y_{t+1}) = c^k(A, Y),$$

given that you can compute

$$c^k(a_t, y_t) = c^k(\mathcal{A}^k(A, Y), Y).$$

(g) As a final step of the numerical analysis, extend your program and simulate the optimal consumption, savings, and labor market behavior of $I = 10^4$ individuals over $T = 10^3$ periods.

(h) What endogenous interest rate

$$r^\star = \alpha \left( \frac{\sum_{i \in I} k_i}{\sum_{i \in I} y_i h_i} \right)^{\alpha - 1} - \delta = \alpha \left( \frac{\sum_{i \in I} a_i}{\sum_{i \in I} y_i h_i} \right)^{\alpha - 1} - \delta, \quad \alpha = 1/3,\ \delta = 5/100,$$

would be consistent with the stationary wealth distribution (the one observed in simulation period $T$)? How would (you don't have to do it, just explain) you solve the same Aiyagari model with an endogenous interest rate wage rate then?

(i) Compared to the transition probability of $\rho = 9/10$ that we considered so far, how does the stationary wealth distribution change if the producitivity shocks are instead i.i.d. over time, $\rho = 1/2$?