Problem Set 2

Exercise 2.1: Neoclassical growth in discrete time: a closed form solution

Consider a version of the neoclassical growth model with Greenwood, Hercowitz, and Huffman (AER, 1988) preferences and 100% physical capital depreciation

$$\max U = \sum_{t=0}^{\infty} \beta^t \frac{\left(C_t - \psi N_t^{\theta}\right)^{1-\sigma} - 1}{1-\sigma}, \quad \theta > 1, \quad \psi > 0, \quad \sigma > 1,$$

subject to the resource constraint

$$K_{t+1} = Y_t - C_t$$
$$Y_t = AK_t^{\alpha} N_t^{1-\alpha}, \quad 0 < \alpha < 1,$$

where $K_0 > 0$ is given. The parameter $0 < \beta < 1$ denotes the representative agent's discount factor, C_t is aggregate consumption, K_t aggregate physical capital, N_t aggregate labor supply, Y_t aggregate production, A total factor productivity in production, α is the aggregate income share from physical capital, θ relates to the Frisch elasticity of labor supply, ψ is a measure for the weight of leisure relative to consumption in the utility function, and σ relates inversely to the intertemporal substitution elasticity of consumption.

(a) State the Lagrangian and derive the first-order conditions for the planner's problem of this economy.

(b) In the competitive equilibrium that corresponds to this planner solution, wages would be given by

$$W_t = (1-\alpha)A(K_t/N_t)^{\alpha}.$$

Use the intra-temporal optimality conditions to show that the labor supply is a function of the wage only and therefore independent of the agents' wealth (The absence of wealth effects comes from the GHH preference specification and makes the model tractable). In addition compute the Frisch elasticity of the labor supply.

(c) Let $\sigma \to 1$ for the rest of this exercise. Derive the consumption Euler equation, then show that $\psi\theta N_t^{\theta} = (1-\alpha)Y_t$ and verify the guess that consumption is proportional to output, $C_t = \mu Y_t$ (where μ is a constant that you will have to determine). Characterize all equilibrium variables in period t as explicit functions of the physical capital stock K_t.

(d) Show that the solution satisfies the transversality condition

$$\lim_{t\to\infty} \beta^t \left(C_t - \psi N_t^{\theta}\right)^{-1} K_{t+1} = 0.$$

(e) Introduce exogenous technical progress into the model

$$K_{t+1} = AK_t^\alpha (X_t N_t)^{1-\alpha} - C_t$$
$$X_{t+1} = (1+\gamma)X_t, \quad X_0 > 0, \gamma > 0.$$

Is there a feasible balanced growth path for this economy? (hint: check whether labor supply N_t (which cannot exceed the constant time endowment) is constant when capital grows at the trend growth rate γ.)

Exercise 2.2: Neoclassical growth in continuous time: a closed form solution

Consider the following version of the neoclassical growth model in continuous time

$$\max U = \int_0^\infty \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t}{b} \right) e^{-\rho t} dt, \quad \rho > 0, \quad \sigma > 1, \quad b > 0,$$

subject to

$$\dot{K}_t = AK_t^\alpha N_t^{1-\alpha} - C_t - \delta K_t, \quad K_0 > 0, \quad 0 < \alpha < 1. \tag{1}$$

C_t is aggregate consumption, N_t aggregate labor supply, K_t aggregate physical capital, A total factor productivity in production, α is the aggregate income share from physical capital, δ is the depreciation rate of physical capital, $1/\sigma$ measures the elasticity of intertemporal substitution of consumption in the utility function, and b scales the disutility from supplying labor.

(a) Write up the current-valued Hamiltonian of this maximization problem and derive the optimality conditions.

(b) Show that the steady-state of this dynamic system is given by

$$C = \frac{[(1-\alpha)bA]^{1/\sigma} (A\alpha)^{\alpha/[\sigma(1-\alpha)]}}{(\rho+\delta)^{\alpha/[\sigma(1-\alpha)]}}$$
$$K = \frac{1}{(1/\alpha - 1)(\rho+\delta) + \rho} \times C$$
$$N = \left(\frac{\alpha A}{\rho+\delta} \right)^{-1/(1-\alpha)} \times K$$
$$\lambda = C^{-\sigma}.$$

(c) Show that Equation (1) together with the optimality conditions imply that

$$(1-1/\alpha)\lambda_t^{-1/\alpha}\dot{\lambda}_t = (1-1/\alpha)(\rho+\delta)\lambda_t^{1-1/\alpha} + (1-\alpha)A[(1-\alpha)Ab]^{1/\alpha-1}$$
$$\dot{K}_t = \left[A[(1-\alpha)Ab]^{1/\alpha-1}\lambda_t^{1/\alpha-1} - \delta \right] K_t - \left(\lambda_t^{1-1/\alpha} \right)^{\alpha/[\sigma(1-\alpha)]}.$$

Or, defining $\mu_t \equiv \lambda_t^{1-1/\alpha}$, $\omega \equiv -(1-1/\alpha)(\rho+\delta)$, and $\psi \equiv A[(1-\alpha)Ab]^{1/\alpha-1}$

$$\dot{\mu}_t = -\omega\mu_t + (1-\alpha)\psi \tag{2}$$
$$\dot{K}_t = [\psi/\mu_t - \delta] K_t - \mu_t^{\alpha/[\sigma(1-\alpha)]}. \tag{3}$$

(d) For the rest of this exercise suppose that by incredible coincidence, $\alpha = \sigma$. Figure out the solution to the generic differential equation (a good option is to visit the website of WolframAlpha, www.wolframalpha.com)

$$\dot{x}_t = a_t x_t + c_t.$$

Then, show that the analytical solution to the above system of differential equations (2), (3) is given by

$$\mu_t = \bar{\mu} + e^{-\omega t}(\mu_0 - \bar{\mu}), \quad \bar{\mu} = (1-\alpha)\psi/\omega$$
$$K_t = \left(\frac{\mu_t}{\mu_0}\right)^{1/(1-\alpha)} e^{(\omega+\rho)t}\left[K_0 - \mu_0^{1/(1-\alpha)}\int_0^t e^{-(\omega+\rho)s}ds\right],$$

conditional on μ_0 and K_0.

(e) Characterize the μ_0 that satisfies the transversality condition as a function of the initial condition K_0.

(f) Show that this model's dynamic solution is therefore given by

$$C_t = [1 + (\mu_0/\bar{\mu} - 1)e^{-\omega t}]^{1/(1-\alpha)} \times C$$
$$K_t = [1 + (\mu_0/\bar{\mu} - 1)e^{-\omega t}]^{1/(1-\alpha)} \times K$$
$$N_t = N.$$

Exercise 2.3: A model of perpetual youth

Consider the Poisson death model (discussed in class, lecture 1, slides 24-25) where agents face a constant probability of survival $0 < \delta < 1$ each period. Agents are not altruistic towards their offsprings, so their preference can be represented by the utility function

$$\max U = \sum_{t=0}^{\infty} (\beta\delta)^t u(c_t),$$

where $u(c)$ denotes the momentary utility from consumption, and β is the subjective discount rate. Agents have an endowment of y_t each period, and they can save (but not borrow) in a one-period bond denoted by b_t which (net) returns r each period. Agents in this economy face a substantial risk, namely, they might die with a positive amount of wealth which - *ex post* - they would have preferred to consume earlier in life.

(a) State an agent's flow budget constraint (conditional on being alive) and the sequential Lagrangian associated with the agent's maximization problem. Then derive the consumption Euler equation. How does the survival probability δ influence the agent's savings behavior?

(b) Consider an insurance company that provides actuarially fair insurance (expected profit from providing the insurance is zero, plus assume that providing insurance has no overhead cost) against passing away with positive wealth. This insurance contract is state-contingent such that agent gets a payoff of zero in case of death, but gets a payment $x > 0$ if they stay alive per unit of insurance. Compute the (conditional on survival) payment x of one unit of insurance.

(c) What is the agent's period-by-period budget constraint if she can buy (but not sell) life-insurance?

(d) State the Lagrangian associated to the the agent's maximization problem and derive the consumption Euler equation for the optimal amount of insurance, a_{t+1} (the Euler equation for the bond remains unchanged).

(e) Does the agent buy the one-period bond and at the same time the insurance in equilibrium? Given that the agent buys insurance, $a_{t+1} > 0$, does the savings behavior of the agent still depend on the probability of survival?

(f) Find the dynamic equilibrium for the case where $a_0 > 0$, $b_0 = 0$, $\beta(1+r) = 1$, $y_t = y > 0$.