The exam consists of five parts: A, B, C, D, and E. All parts have the same weight (20%).

## Part A: Preliminaries (20 %)

a) Explain in your own words why the Gorman form is sufficient for the economy to admit a representative agent.

Assume that the individual maximizes $U(c_1, c_2) = c_1^{\alpha} c_2^{1-\alpha}$ subject to the budget constraints $y = p_1 c_1 + p_2 c_2$.

b) Show that the solution to this problem implies Marshallian demand functions that are linear in income (linear Engel curves) and therefore satisfy the Gorman form.

Assume now that the individual maximizes $U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma}}{1-\gamma}$ subject to $c_1 + a = y_1$ and $c_2 = a + y_2$.

c) Show that the solution to this problem implies Marshallian demand functions that are linear in income and therefore satisfy the Gorman form.

d) Suppose now that we also include a borrowing constraint $a \geq 0$. Show that the solution no longer implies Marshallian demand functions that are linear in income. Does an economy with borrowing constraints necessarily admit a representative agent?

## Part B: The Neoclassical Growth Model (20 %)

We are going to use the Neoclassical Growth model to investigate how the long-run equilibrium in an economy responds to changes in fundamental parameters. The model is

$$\frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta - \rho$$
$$\dot{k}(t) = k(t)^{\alpha} - (n+\delta)k(t) - c(t)$$

where we assume that $\alpha = 0.5$, $\delta = 0.05$, $\rho = 0.05$, and $n = 0.05$.

a) Compute the steady state values $k^*$ and $c^*$, both algebraically and numerically.

b) Assume that $k_0 < k^*$. Show how the economy moves toward the steady state and explain the intuition.

c) Assume that the economy is at a steady state consistent with $n = 0.05$ and that the fertility rate in the country suddenly falls permanently to $n = 0$. Show how the economy moves toward its new steady state and explain the intuition.

d) Assume again that the economy is at a steady state consistent with $n = 0.05$. Assume now that the economy suddenly experiences a change in the production structure such that $\alpha$ decreases from 0.5 to 0.4. Show how the economy moves toward its new steady state and explain the intuition.

## Part C: Real Business Cycle (20 %)

Assume that in every period $t$ observations of GDP, $Y_t$, are generated by the following equations:

$$Y_t = Y_0 \exp(\varepsilon_t), \qquad \varepsilon_t = g + \varepsilon_{t-1} + u_t$$

where $u_t \sim N(0, \sigma)$.

a) Describe the difference between a data series that has a *deterministic* vs. a *stochastic* trend. Does the GDP series $Y_t$ have a deterministic or a stochastic trend?

b) Show how you would estimate the cyclical component of the GDP series $Y_t$.

Consider the resource constraint of an economy with a net population growth rate $g_N$ and a net rate of technological progress given by $g_A$. With a Cobb-Douglas production function this constraint is given by:

$$K_{t+1} + C_t = Z_t K_t^{\alpha} (A_t N_t h_t)^{1-\alpha} + (1-\delta) K_t \tag{1}$$

where the notation for variables and parameters are the same as in the lecture notes on the RBC model.

c) Define what we mean by a *balanced growth path*.

d) Show how to express the resource constraint in equation (1) in terms of variables that are constant on the balanced growth path.

In one version of the model presented in class (when we ignored growth and set both $g_A = g_N = 0$), the set of equations to solve for equilibrium included the following:

| | |
|---|---|
| Production: | $y_t = Z_t k_t^{\alpha} h_t^{1-\alpha}$ |
| Labor-leisure trade-off: | $\dfrac{\theta c_t}{1-h_t} = (1-\alpha) Z_t k_t^{\alpha} h_t^{-\alpha} = w_t$ |
| Resource constraint: | $k_{t+1} + c_t = y_t + (1-\delta) k_t$ |
| Wage: | $w_t = (1-\alpha) \dfrac{y_t}{h_t}$ |
| Interest rate: | $R_t = \alpha \dfrac{y_t}{k_t} + 1 - \delta$ |

e) Show how to derive the log-linearized versions of the wage equation, the labor-leisure trade-off, and the resource constraint.

## Part D: The New-Keynesian Model (20 %)

We will use the New Keynesian model to see how the central bank should optimally respond to a demand shock. We assume that the model is

$$\begin{aligned}
\pi_t &= \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa x_t \\
x_t &= \mathbb{E}_t\{x_{t+1}\} - (i_t - \mathbb{E}_t\{\pi_{t+1}\}) + d_t \\
d_t &= \rho d_{t-1} + \varepsilon_t \\
\varepsilon_t &\sim N(0, \sigma)
\end{aligned}$$

where $\pi$ is inflation, $x$ is the output gap, $i$ is the nominal interest rate, and $d$ is a discount rate shock. $\kappa > 0$, $\rho > 0$, and $\sigma > 0$ are known parameters.

a) Assume that the central bank's interest rate rule is $i_t = 0$. Compute how the economy responds to the discount rate shock using the method of undetermined coefficients.

b) Explain with words how a positive discount rate shock affects the economy when the central bank does not respond to the shock.

Assume now that the central bank's loss function is $L_t = \mathbb{E}_t \sum_{k=0}^{\infty} \beta^k [\pi_{t+k}^2 + \lambda x_{t+k}^2]$.

c) Solve for the optimal policy under discretion and interpret the first-order conditions (the 'leaning against the wind condition').

d) Show that the optimal interest rate policy is $i_t = d_t$. Explain why it looks like that.

## Part E: Borrowing Constraints and Investments (20 %)

Suppose we have an economy with two types of agents: savers and entrepreneurs. Entrepreneurs have a discount factor $\beta^e < \beta^s$, where $\beta^s$ is the discount factor of the savers. Both agents invest in machines and equipment $m_t^i$ with a price $p_t$ for $i \in \{e, s\}$. Machines and equipment are in fixed supply. Both agents use these machines and equipment to produce. The production function of entrepreneurs is given by $y_t = am_{t-1}^e$, whereas savers has a production function $y_t^s = \left(m_{t-1}^s\right)^\alpha$, with $\alpha < 1$. As a result of the different discount factors, entrepreneurs will be borrowers in equilibrium. The interest rate in the economy is given by

$$R = \frac{1}{\beta^s}.$$

Due to limited enforcement, entrepreneurs are subject to a borrowing constraint

$$b_t R_t < p_{t+1} m_t^e$$

a) Suppose $\beta^s$ increases, while nothing happens to the price of machines and equipment. Can this affect productive efficiency in the economy? How? Why?

Suppose now that the production technology of the savers is risky, i.e., that the realized return is $y_t^s = z_t + \left(m_{t-1}^s\right)^\alpha$ where $z_t \sim \mathcal{N}(0, \sigma_z)$. In addition, suppose that savers get a stochastic endowment $x_t \sim \mathcal{N}(0, \sigma_x)$ each period. Moreover, since the entrepreneurs are borrowing constrained, the value of machines and equipment is fully determined by the savers' valuation.

b) Explain intuitively (no math necessary) what determines the savers' valuation of machines and equipment. How does it depend on $cov(x_t, z_t)$? What is the intuition?

Consider a model where agents have utility over consumption in period $T$ $u(c) = 1 - \dfrac{1}{c_T}$. There are three time periods: $T = 0, 1, 2$. All agents are endowed with one unit of a homogeneous good. In period 1, agents learn whether they are 'early' or 'late' types, where early types have utility over consumption in period 1 and late types have utility over consumption in period 2, so that

$$u^{\text{early}} = 1 - \frac{1}{c_1}$$
$$u^{\text{late}} = 1 - \frac{1}{c_2}$$

The population fraction of type 'early' is $t = 0.25$. There is no discounting. Suppose that there are three investment opportunities

1. Storage: yields 1 unit in $T = 1$ and 1 unit in $T = 2$
2. Investment: yields $R$ units in $T = 2$, but can be liquidated in period $T = 1$ to yield 1 unit.
3. Deposits: yields $r_1 = 1.3$ in $T = 1$ and $r_2 = 1.8$ in $T = 2$

c) Show that the deposits contract yields the highest expected utility in $T = 0$. Explain briefly the intuition.

Consider the same model as in the previous problem (c), but where there is a fourth alternative asset 'Bitcoin.' The returns from holding a Bitcoin is

$$r_1^b = \begin{cases} 1.5 & \text{with probability } p \\ 1 & \text{otherwise} \end{cases} \qquad \text{and } r_2^b = 2.$$

d) How high does $p$ have to be for the agents to prefer Bitcoin over deposits?