The exam consists of five parts: A, B, C, D, and E. All parts have the same weight (20%).

Part A: Preliminaries (20 %)

a) Explain in your own words why the Gorman form is sufficient for the economy to admit a representative agent.

Solution: the Gorman form implies that behavior scales with income such that the income distribution does not matter for aggregate outcomes.

Assume that the individual maximizes $U(c_1, c_2) = c_1^{\alpha} c_2^{1-\alpha}$ subject to the budget constraints $y = p_1 c_1 + p_2 c_2$.

b) Show that the solution to this problem implies Marshallian demand functions that are linear in income (linear Engel curves) and therefore satisfy the Gorman form.

Solution: the Marshallian demand functions are $c_1 = \frac{\alpha}{p_1} y$ *and* $c_2 = \frac{1-\alpha}{p_2} y$.

Assume now that the individual maximizes $U(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma}}{1-\gamma}$ subject to $c_1 + a = y_1$ and $c_2 = a + y_2$.

c) Show that the solution to this problem implies Marshallian demand functions that are linear in income and therefore satisfy the Gorman form.

Solution: the Marshallian demand functions are $c_1 = c_2 = \frac{1}{2}(y_1 + y_2)$.

d) Suppose now that we also include a borrowing constraint $a \geq 0$. Show that the solution no longer implies Marshallian demand functions that are linear in income. Does an economy with borrowing constraints necessarily admit a representative agent?

Solution: the Marshallian demand functions are $c_1 = \begin{cases} y_1 & \text{if } y_1 < y_2 \\ \frac{1}{2}(y_1 + y_2) & \text{if } y_1 \geq y_2 \end{cases}$ *and* $c_2 = \begin{cases} y_2 & \text{if } y_1 < y_2 \\ \frac{1}{2}(y_1 + y_2) & \text{if } y_1 \geq y_2 \end{cases}$. *The demand functions are piecewise linear, but not linear in income. Hence, the household problem does not satisfy the Gorman form and the economy does not necessarily admit a representative agent.*

Part B: The Neoclassical Growth Model (20 %)

We are going to use the Neoclassical Growth model to investigate how the long-run equilibrium in an economy responds to changes in fundamental parameters. The model is

$$\frac{\dot{c}(t)}{c(t)} = \alpha k(t)^{\alpha-1} - \delta - \rho$$
$$\dot{k}(t) = k(t)^{\alpha} - (n+\delta)k(t) - c(t)$$

where we assume that $\alpha = 0.5$, $\delta = 0.05$, $\rho = 0.05$, and $n = 0.05$.

a) Compute the steady state values k^* and c^*, both algebraically and numerically.

Solution: $k^* = \left(\frac{\alpha}{\delta+\rho}\right)^{1/(1-\alpha)} = (0.5/0.1)^2 = 25$; $c^* = (k^*)^{\alpha} - (n+\delta)k^* = \sqrt{25} - 0.1 * 25 = 2.5.$

b) Assume that $k_0 < k^*$. Show how the economy moves toward the steady state and explain the intuition.

Solution: For a given k^0, there is one and only one c that is consistent with the path to the new steady state. First, consumption has to be lower than the locus consistent with $\dot{k} = 0$ to make sure that capital grows. Further, since $k_0 < k^*$, consumption is growing. At each point of the path, both capital and consumption has to grow. The path illustrated in Figure 1 is one example of such a path.

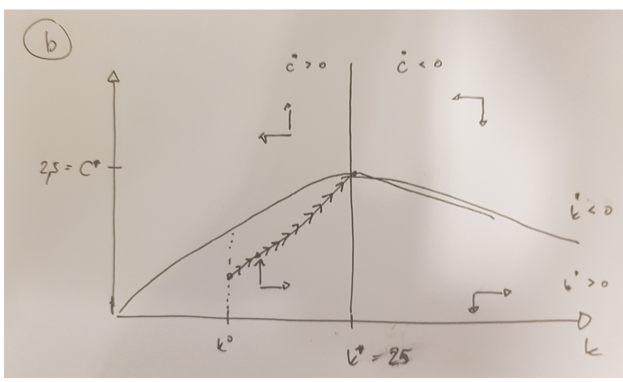

Figure 1: 2b

c) Assume that the economy is at a steady state consistent with $n = 0.05$ and that the fertility rate in the country suddenly falls permanently to $n = 0$. Show how the economy moves toward its new steady state and explain the intuition.

Solution: with a change in n, nothing happens with k^*. But $c^* = 5 - 0.05 * 5 = 3.75$. The economy immediately jumps to the new steady state with $k^* = 25$ and $c^* = 3.75$ as in Figure 2. Why? The population suddenly stops growing so that the need to save to keep capital per person stable has disappeared. Households therefore increase consumption per person. The change is immediate.

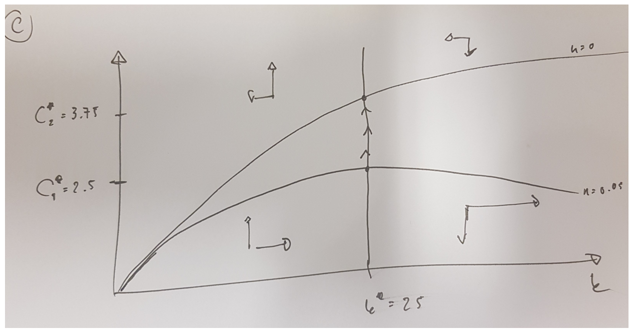

Figure 2: 2c

d) Assume again that the economy is at a steady state consistent with $n = 0.05$. Assume now that the economy suddenly experiences a change in the production structure such that α decreases from 0.5 to 0.4. Show how the economy moves toward its new steady state and explain the intuition.

Solution: A change in α is making capital less productive everywhere and at the margin. There should therefore be less capital in the economy in the new steady state. While reducing capital, the households consumes a little more than what is consistent with the new steady state. The numerical values for the new steady state are $k^* = (0.4/0.1)^{1/(1-0.4)} \approx 10.08$ and $c^* \approx 1.51$. Figure 3 illustrates an example path.

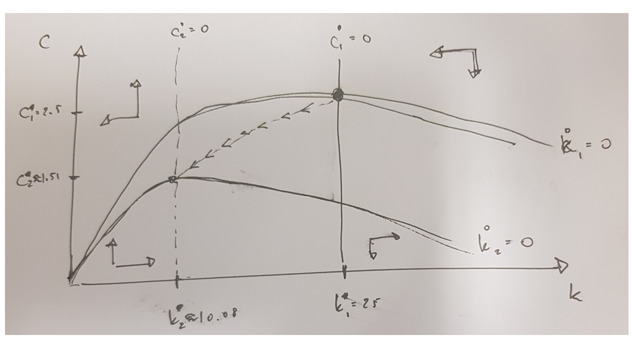

Figure 3: 2d

Part C: Real Business Cycle (20 %)

Assume that in every period t observations of GDP, Y_t, are generated by the following equations:

$$Y_t = Y_0 \exp(\varepsilon_t), \qquad \varepsilon_t = g + \varepsilon_{t-1} + u_t$$

where $u_t \sim N(0, \sigma)$.

a) Describe the difference between a data series that has a *deterministic* vs. a *stochastic* trend. Does the GDP series Y_t have a deterministic or a stochastic trend?

Solution: A data series has a deterministic trend if the trend grows at the same rate every period. That is, the trend has a constant growth rate. A data series has a stochastic trend if there are shocks to the trend component making it change over time. For the GDP series in this question we have:

$$\begin{aligned}
\epsilon_t &= g + \epsilon_{t-1} + u_t \\
&= g + g + \epsilon_{t-2} + u_t + u_{t-1} \\
&= g \cdot 3 + \epsilon_{t-3} + u_t + u_{t-1} + u_{t-2} \\
&\dots \\
&= g \cdot t + \sum_{j=1}^{t-1} u_{t-j} + \epsilon_0
\end{aligned}$$

Hence, the trend is not constant, but is influenced by shocks to the series, and this GDP series has a stochastic trend.

b) Show how you would estimate the cyclical component of the GDP series Y_t.

Solution: We estimate the cyclical component by taking the first difference of the log of the series:

$$\begin{aligned}
\log Y_{t+1} - \log Y_t &= \log Y_0 + \epsilon_{t+1} - \log Y_0 - \epsilon_t \\
&= \epsilon_{t+1} - \epsilon_t = g + u_{t+1}
\end{aligned}$$

We can then estimate g by taking the mean of these first differences. The shocks u_{t+1} will then cancel out as they have a mean of zero, and thus we obtain the cyclical components u_{t+1}.

Consider the resource constraint of an economy with a net population growth rate g_N and a net rate of technological progress given by g_A. With a Cobb-Douglas production function this constraint is given by:

$$K_{t+1} + C_t = Z_t K_t^{\alpha} (A_t N_t h_t)^{1-\alpha} + (1-\delta) K_t \tag{1}$$

where the notation for variables and parameters are the same as in the lecture notes on the RBC model.

c) Define what we mean by a *balanced growth path*.

Solution: By the balanced growth path we mean an equilibrium of the model where all variables are either constant or grow at the same rate when there are no shocks hitting the economy. The growth rate of all growing variables are determined by population growth and technological progress on the balanced growth path. (Variables defined in per-capita terms grow at the rate of technological progress along the balanced growth path. Variables rescaled by both the level of population and the level of technology are constant along the balanced growth path.)

d) Show how to express the resource constraint in equation (1) in terms of variables that are constant on the balanced growth path.

Solution: We first express all the growing variables in per-capita terms. That means we divide the equation by N_t and define variable $x_t = X_t/N_t$. Then we get:

$$\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} + \frac{C_t}{N_t} = Z_t \left(\frac{K_t}{N_t}\right)^{\alpha} (A_t h_t)^{1-\alpha} + (1-\delta) \frac{K_t}{N_t}$$
$$\Rightarrow \; k_{t+1}(1+g_N) + c_t = Z_t k_t^{\alpha} (A_t h_t)^{1-\alpha} + (1-\delta) k_t$$

Next we divide by the level of technology A_t and define variable $\tilde{x}_t = \frac{X_t}{A_t N_t}$:

$$\frac{k_{t+1}}{A_{t+1}} \frac{A_{t+1}}{A_t}(1+g_N) + \frac{c_t}{A_t} = Z_t \left(\frac{k_t}{A_t}\right)^{\alpha} (h_t)^{1-\alpha} + (1-\delta) \frac{k_t}{A_t}$$
$$\tilde{k}_{t+1}(1+g_A)(1+g_N) + \tilde{c}_t = Z_t \tilde{k}_t^{\alpha} h_t^{1-\alpha} + (1-\delta)\tilde{k}_t.$$

On the balanced growth path there are no shocks hitting the economy, so $Z_t = 1$. Then all the variables in the equation above are constant, and we have the following equation for the resource constraint on the balanced growth path:

$$\tilde{k}(1+g_A)(1+g_N) + \tilde{c} = \tilde{k}^{\alpha} h^{1-\alpha} + (1-\delta)\tilde{k}.$$

In one version of the model presented in class (when we ignored growth and set both

$g_A = g_N = 0$), the set of equations to solve for equilibrium included the following:

$$\begin{aligned}
\text{Production:} \quad & y_t = Z_t k_t^{\alpha} h_t^{1-\alpha} \\
\text{Labor-leisure trade-off:} \quad & \frac{\theta c_t}{1-h_t} = (1-\alpha) Z_t k_t^{\alpha} h_t^{-\alpha} = w_t \\
\text{Resource constraint:} \quad & k_{t+1} + c_t = y_t + (1-\delta) k_t \\
\text{Wage:} \quad & w_t = (1-\alpha)\frac{y_t}{h_t} \\
\text{Interest rate:} \quad & R_t = \alpha \frac{y_t}{k_t} + 1 - \delta
\end{aligned}$$

e) Show how to derive the log-linearized versions of the wage equation, the labor-leisure trade-off, and the resource constraint.

Solution: The wage equation:

$$1 \cdot w \cdot \hat{w}_t = (1-\alpha)\frac{1}{h} \cdot y \cdot \hat{y}_t - (1-\alpha)\frac{y}{h^2} \cdot h \cdot \hat{h}_t \tag{2}$$

$$\Rightarrow \; w \cdot \hat{w}_t = (1-\alpha)\frac{y}{h}\left(\hat{y}_t - \hat{h}_t\right) \tag{3}$$

$$\Rightarrow \; \hat{w}_t = \hat{y}_t - \hat{h}_t \tag{4}$$

where the last line uses the fact that the equation must hold along the balanced growth path (or steady state) so that $w = (1-\alpha)y/h$.

The labor-leisure trade-off:

$$\begin{aligned}
\theta \frac{1}{1-h} \cdot c \cdot \hat{c}_t - \frac{\theta c}{(1-h)^2} \cdot (-1) \cdot h \cdot \hat{h}_t &= 1 \cdot w \cdot \hat{w}_t \\
\Rightarrow \; \frac{\theta c}{1-h}\left(\hat{c}_t + \frac{h}{1-h}\hat{h}_t\right) &= w \hat{w}_t \\
\Rightarrow \; \hat{c}_t + \frac{h}{1-h}\hat{h}_t &= \hat{w}_t
\end{aligned}$$

where again, the last line uses that the equation must hold along the balanced growth path.

The resource constraint:

$$\begin{aligned}
1 \cdot k \cdot \hat{k}_{t+1} + 1 \cdot c \cdot \hat{c}_t &= 1 \cdot y \cdot \hat{y}_t + (1-\delta) \cdot k \cdot \hat{k}_t \\
\Rightarrow \; k\hat{k}_{t+1} + c\hat{c}_t &= y\hat{y}_t + (1-\delta)k\hat{k}_t.
\end{aligned}$$

Part D: The New-Keynesian Model (20 %)

We will use the New Keynesian model to see how the central bank should optimally respond to a demand shock. We assume that the model is

$$\begin{aligned}
\pi_t &= \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa x_t \\
x_t &= \mathbb{E}_t\{x_{t+1}\} - (i_t - \mathbb{E}_t\{\pi_{t+1}\}) + d_t \\
d_t &= \rho d_{t-1} + \varepsilon_t \\
\varepsilon_t &\sim N(0, \sigma)
\end{aligned}$$

where π is inflation, x is the output gap, i is the nominal interest rate, and d is a discount rate shock. $\kappa > 0$, $\rho > 0$, and $\sigma > 0$ are known parameters.

a) Assume that the central bank's interest rate rule is $i_t = 0$. Compute how the economy responds to the discount rate shock using the method of undetermined coefficients.

Solution: $x_t = \frac{1-\beta\rho}{(1-\beta\rho)(1-\rho)-\kappa\rho} d_t$ *and* $\pi_t = \frac{\kappa}{(1-\beta\rho)(1-\rho)-\kappa\rho} d_t$.

b) Explain with words how a positive discount rate shock affects the economy when the central bank does not respond to the shock.

Solution: a discount rate shock first raises current output. This raises prices through the Phillips curve. Since prices increases, the real interest rate falls, thus further stimulating output. This effect can be observed from $x_t > d_t$ *always. As the shock eventually dies out, so does the effects on output and inflation.*

Assume now that the central bank's loss function is $L_t = \mathbb{E}_t \sum_{k=0}^{\infty} \beta^k [\pi_{t+k}^2 + \lambda x_{t+k}^2]$.

c) Solve for the optimal policy under discretion and interpret the first-order conditions (the 'leaning against the wind condition').

Solution: optimal policy under discretion implies $\lambda x_t = -\kappa \pi_t$. *Marginal cost of leaning in terms of output deviations equals marginal benefits in terms of reducing inflation deviations.*

d) Show that the optimal interest rate policy is $i_t = d_t$. Explain why it looks like that.

Solution: since the discount rate shock does not imply any trade-off for monetary policy, optimal monetary policy is $\lambda x_t = -\kappa \pi_t = 0$. *This policy can be implemented as long as* $x_t = 0$, *which is consistent with* $i_t = d_t$. *Hence, optimal monetary policy implies fully canceling out any discount rate shocks.*

Part E: Borrowing Constraints and Investments (20 %)

Suppose we have an economy with two types of agents: savers and entrepreneurs. Entrepreneurs have a discount factor $\beta^e < \beta^s$, where β^s is the discount factor of the savers. Both agents invest in machines and equipment m_t^i with a price p_t for $i \in \{e, s\}$. Machines and equipment are in fixed supply. Both agents use these machines and equipment to produce. The production function of entrepreneurs is given by $y_t = am_{t-1}^e$, whereas savers has a production function $y_t^s = \left(m_{t-1}^s\right)^\alpha$, with $\alpha < 1$. As a result of the different discount factors, entrepreneurs will be borrowers in equilibrium. The interest rate in the economy is given by

$$R = \frac{1}{\beta^s}.$$

Due to limited enforcement, entrepreneurs are subject to a borrowing constraint

$$b_t R_t < p_{t+1} m_t^e$$

a) Suppose β^s increases, while nothing happens to the price of machines and equipment. Can this affect productive efficiency in the economy? How? Why?

Solution: Yes, an increase in β^s can increase productive efficiency. $\beta^s \uparrow \Rightarrow R \downarrow$. This relaxes the borrowing constraint. Relaxation of the borrowing constraint reallocates machines and equipment to the entrepreneurs. If the marginal product of capital is high for the entrepreneurs relative to savers, this will increase production.

Suppose now that the production technology of the savers is risky, i.e., that the realized return is $y_t^s = z_t + \left(m_{t-1}^s\right)^\alpha$ where $z_t \sim \mathcal{N}(0, \sigma_z)$. In addition, suppose that savers get a stochastic endowment $x_t \sim \mathcal{N}(0, \sigma_x)$ each period. Moreover, since the entrepreneurs are borrowing constrained, the value of machines and equipment is fully determined by the savers' valuation.

b) Explain intuitively (no math necessary) what determines the savers' valuation of machines and equipment. How does it depend on $cov(x_t, z_t)$? What is the intuition?

Solution: The student should discuss - without necessarily using math - how the value of machines and equipment is the discounted value of future payoffs. The student should discuss thoroughly how the covariance between the productivity parameter z_t and the endowment x_t determines the valuation, i.e.,

1. $cov(x_t, z_t) = 0 \Rightarrow p_t = \beta^s \times \mathbb{E}[y_{+1} + p_{t+1}]$
2. $cov(x_t, z_t) > 0 \Rightarrow$ *lower p_t compared to the first case.*

3. *$cov(x_t, z_t) < 0 \Rightarrow$ higher p_t compared to the first case.*

The intuition is that $cov(x_t, z_t)$ determines the degree of consumption insurance savers gets from owning machines and equipment. If machines and equipment generates consumption insurance $cov(x_t, z_t) < 0$, it is more valuable.

Consider a model where agents have utility over consumption in period T $u(c) = 1 - \frac{1}{c_T}$. There are three time periods: $T = 0, 1, 2$. All agents are endowed with one unit of a homogeneous good. In period 1, agents learn whether they are 'early' or 'late' types, where early types have utility over consumption in period 1 and late types have utility over consumption in period 2, so that

$$
\begin{aligned}
u^{\text{early}} &= 1 - \frac{1}{c_1} \\
u^{\text{late}} &= 1 - \frac{1}{c_2}
\end{aligned}
$$

The population fraction of type 'early' is $t = 0.25$. There is no discounting. Suppose that there are three investment opportunities

1. Storage: yields 1 unit in $T = 1$ and 1 unit in $T = 2$
2. Investment: yields R units in $T = 2$, but can be liquidated in period $T = 1$ to yield 1 unit.
3. Deposits: yields $r_1 = 1.3$ in $T = 1$ and $r_2 = 1.8$ in $T = 2$

c) Show that the deposits contract yields the highest expected utility in $T = 0$. Explain briefly the intuition.

Solution: Just compute the expected utilities:

$$
\begin{aligned}
\mathbb{E}(u)^{\text{store}} &= 0.25 \times 0 + 0.75 \times 0 = 0 \\
\mathbb{E}(u)^{\text{invest}} &= 0.25 \times 0 + 0.75 \times (1 - \frac{1}{2}) = \frac{3}{8} \\
\mathbb{E}(u)^{\text{deposit}} &= 0.25 \times \left(1 - \frac{1}{1.3}\right) + 0.75 \times \left(1 - \frac{1}{1.8}\right) = 0.391
\end{aligned}
$$

Since $0.391 > \frac{3}{8} = 0.375$, the expected utility from investing in the deposit contract is highest. This is because the consumption profile under deposit contract is higher compared to storage and smoother compared to investment.

Consider the same model as in the previous problem (c), but where there is a fourth alternative asset 'Bitcoin.' The returns from holding a Bitcoin is

$$r_1^b = \begin{cases} 1.5 & \text{with probability } p \\ 1 & \text{otherwise} \end{cases} \qquad \text{and } r_2^b = 2.$$

d) How high does p have to be for the agents to prefer Bitcoin over deposits?

Solution: The expected utility from holding Bitcoin is $\mathbb{E}(u)^{Bitcoin} = 0.25 \times \left(p \times \left(1 - \frac{1}{1.5}\right)\right) + 0.75 \times \left(1 - \frac{1}{2}\right)$. *To prefer Bitcoin, the p has to high enough such that expected utility from holding Bitcoing is greater than 0.391. This p is 0.192. Hence, if* $p > 0.192$, *the agents prefer Bitcoin over deposits.*