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Exam
Advanced Macroeconomic Theory ECON 5300

Good Luck!

1 Real Business Cycles

Consider a Real Business Cycle Economy, which is characterized by the solution to the social planner problem:

$$\max_{c_t, l_t} E\left(\sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + 2\log(1 - l_t)\right]\right)$$

subject to

$$\begin{aligned} c_t + i_t &= y_t \\ k_{t+1} &= i_t + (1-\delta)k_t \\ y_t &= k_t^{\alpha} l_t^{1-\alpha} \\ c_t &\geq 0, l_t \in [0,1], k_0 \text{ given} \end{aligned}$$

Note that there is no technological progress and no population growth.

1. Derive the first-order conditions of the social planner problem

   Solution:

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t \\ R_t &= \alpha k_t^{\alpha-1} l_t^{1-\alpha} + (1-\delta) \\ w_t &= (1-\alpha) k_t^{\alpha} l_t^{-\alpha} \\ w_t &= 2\frac{c_t}{1-l_t} \\ c_t^{-1} &= E_t\left[\beta c_{t+1}^{-1} R_{t+1}\right] \end{aligned}$$

   + transversality condition

2. Derive the steady-state relationships

   Solution:

$$\begin{aligned} \bar{c} + \bar{k} &= \bar{k}^{\alpha} \bar{l}^{1-\alpha} + (1-\delta)\bar{k} \\ \bar{w} &= (1-\alpha)\bar{k}^{1-\alpha} \bar{l}^{-\alpha} \\ \bar{w} &= 2\frac{\bar{c}}{1-\bar{l}} \\ \bar{R} &= \alpha \bar{k}^{\alpha-1} + (1-\delta) \\ 1 &= \beta \bar{R} \Leftrightarrow \bar{R} = \frac{1}{\beta} \end{aligned}$$

3. Calibrate the model (at quarterly frequency), that is determine the values α, β and δ, using the following targets:

   - The quarterly marginal product of capital (before depreciation) is 5%.
   - The capital income share is 50%.
   - Assume that $\frac{I}{Y} = 0.3$, where both I and Y are quarterly data.

   Solution:

   - α is given by the capital share in total output

$$0.5 = s^k \equiv \frac{rk}{Y} = \frac{k\alpha k^{\alpha-1} l^{1-\alpha}}{y} = \alpha$$

   - Capital return and Budget constraint:

$$\begin{aligned} \bar{R} &= \alpha\frac{\bar{y}}{\bar{k}} + (1-\delta) \\ \alpha\frac{\bar{y}}{\bar{k}} &= 0.05 \\ \frac{\bar{y}}{\bar{k}} &= \frac{0.05}{0.5} = 0.1 \\ \bar{k} &= (1-\delta)\bar{k} + \bar{I} \\ \delta &= \frac{\bar{I}}{\bar{k}} = \frac{\bar{I}}{\bar{y}}\frac{\bar{y}}{\bar{k}} = 0.3 * 0.1 = 0.03 \end{aligned}$$

   - The Euler-Equation simplifies to

$$1 = \beta\left(\alpha\frac{\tilde{y}}{\tilde{k}} + 1 - \delta\right) = \beta(1 + 0.05 - 0.03)$$

$$\beta = \frac{1}{1.02} \approx 0.98$$

4. What is the capital/output ratio in this economy. Can you recalibrate the economy to obtain a capital/output ratio of 1 and at the same time still match the three targets above. If yes, how? If not, why not?

   Solution:

The capital output ratio, $\frac{\bar{k}}{\bar{y}} = 10$. If $\frac{\bar{k}}{\bar{y}} = 1$ then $\delta = \frac{\bar{I}}{\bar{y}} = 0.3$. $\alpha = 0.5$ as before. But then

$$\alpha \frac{\bar{y}}{\bar{k}} = \alpha = 0.5 \neq 0.05$$

So the three targets cannot be matched.

2 Complete Markets and Asset Pricing

Consider an economic environment with complete markets. Assume that households indexed by $i \in I$ trade all state-contingent claims at time 0 such that a household's maximization problem is of the following form:

$$\max_{c_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t^0(s^t) \frac{c_t^i(s^t)^{1-\gamma}}{1-\gamma},$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) \left[c_t^i(s^t) - y_t^i(s^t) \right] = 0 \quad \forall i \in I.$$

(a) Define an Arrow-Debreu security. Show that the Arrow-Debreu price $q_t^0(s^t)$ is a function of aggregate consumption. Proceed in the following steps: (i) formulate the Langrangean ii) derive the Euler equation of an individual household i, taking as given prices, $q_t^0(s^t)$, and (iii) substitute individual consumption for aggregate consumption

Solution:

The Lagrangian for individual i reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) \frac{c_t^i(s^t)^{1-\gamma}}{1-\gamma} + \lambda^i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) \left[y_t^i(s^t) - c_t^i(s^t) \right].$$

The optimality conditions of individual i with respect to consumption, $c_t^i(s^t)$, read

$$0 = \frac{\partial \mathcal{L}}{\partial c_t^i(s^t)} = \beta^t \pi_t^0(s^t) c_t^i(s^t)^{-\gamma} - \lambda^i q_t^0(s^t), \quad \forall s^t \in S^t. \tag{1}$$

The same optimality conditions hold for any other individual j, such that

$$\left(\frac{c_t^j(s^t)}{c_t^i(s^t)} \right)^{\gamma} = \lambda^i / \lambda^j \quad \Leftrightarrow \quad c_t^j(s^t) = \left(\lambda^i / \lambda^j \right)^{1/\gamma} c_t^i(s^t), \quad \forall s^t \in S^t.$$

Aggregate consumption over all individuals j

$$C_t(s^t) \equiv \sum_{j \in I} c_t^j(s^t) = \sum_{j \in I} \left(\lambda^i / \lambda^j \right)^{1/\gamma} c_t^i(s^t) = c_t^i(s^t) \left(\lambda^i \right)^{1/\gamma} \sum_{j \in I} \left(\lambda^j \right)^{-1/\gamma}, \quad \forall s^t \in S^t,$$

such that the consumption level of individual i can be written in terms of aggregate consumption, $C_t(s^t)$, as

$$c_t^i(s^t) = C_t(s^t)\left(\lambda^i\right)^{-1/\gamma}\left(\sum_{j\in I}\left(\lambda^j\right)^{-1/\gamma}\right)^{-1}, \quad \forall s^t \in S^t. \tag{2}$$

Combine Equation (2) and Equation (1) to substitute out individual consumption

$$\begin{aligned}\lambda^i q_t^0(s^t) &= \beta^t \pi_t(s^t)\left(C_t(s^t)\left(\lambda^i\right)^{-1/\gamma}\left(\sum_{j\in I}\left(\lambda^j\right)^{-1/\gamma}\right)^{-1}\right)^{-\gamma}, \quad \forall s^t \in S^t \\ &= \beta^t \pi_t(s^t) C_t(s^t)^{-\gamma}\lambda^i\left(\sum_{j\in I}\left(\lambda^j\right)^{-1/\gamma}\right)^{\gamma}.\end{aligned}$$

Thus, the individual specific multiplier λ^i cancels out and the Arrow-Debreu prices are given by

$$q_t^0(s^t) = \beta^t \pi_t^0(s^t) C_t(s^t)^{-\gamma}\left(\sum_{j\in I}\left(\lambda^j\right)^{-1/\gamma}\right)^{\gamma}, \quad \forall s^t \in S^t.$$

Finally, we can normalize one price. Let us normalize with respect to the Arrow-Debreu price for the initial state, $q_0^0(s^0) = 1$ (note also that the initial state is deterministic, $\pi_0(s^0) = 1$), such that the final expression is

$$q_t^0(s^t) = \frac{q_t^0(s^t)}{q_0^0(s)} = \beta^t \pi_t^0(s^t)\left(\frac{C_t(s^t)}{C_0(s^0)}\right)^{-\gamma}, \quad \forall s^t \in S^t. \tag{3}$$

Use this price in the individual Euler equation of Equation (1) to yield

$$\beta^t \pi_t(s^t) c_t^i(s^t)^{-\gamma} = \lambda^i q_t^0(s^t) = \lambda^i \beta^t \pi_t^0(s^t)\left(\frac{C_t(s^t)}{C_0(s^0)}\right)^{-\gamma}, \quad \forall s^t \in S^t,$$

which can be written as

$$c_t^i(s^t)^{-\gamma} = \lambda^i\left(\frac{C_t(s^t)}{C_0(s^0)}\right)^{-\gamma}, \quad \forall s^t \in S^t.$$

Now, consider the ratio of marginal utilities across two consecutive histories of state realizations, s^t, and s^{t+1}, for individual i,

$$\left(\frac{c_{t+1}^i(s^{t+1})}{c_t^i(s^t)}\right)^{\gamma} = \left(\frac{C_{t+1}(s^{t+1})}{C_t(s^t)}\right)^{\gamma} = \left(\frac{Y_{t+1}(s^{t+1})}{Y_t(s^t)}\right)^{\gamma}, \tag{4}$$

thus, the individual consumption growth is that same as aggregate consumption growth. The last equality follows from the fact that that aggregate consumption in each state s^t has to be equal to the aggregate endowment in that state (aggregate resource constraint),

$$C_t(s^t) = Y_t(s^t) \equiv \sum_{i\in I} y_t^i(s^t). \tag{5}$$

(b) Assume the existence of a representative household. Are the following statements correct? (i) "If we know the wealth holding of an individual household, we also know the wealth holdings of all other households." (ii) "The wealth distribution matters for the aggregate decision of the household sector."

Solution:

Equation (4) which characterizes individual consumption growth is the relevant to answer this question.

(i) This statement is not correct.

(ii) This statement is not correct.

Suppose from now that individuals also make a labor supply choice, $h_t^i(s^t)$ (hours per period), and face an individual-specific but fixed wage w^i. Thus, labor income each period is $y_t^i(s^t) = w^i h_t^i(s^t)$. Assume that preferences are

$$u(c,h) = \log(c) - (1-h)^{(-1)},$$

such that the objective function of the household reads

$$\max_{c_t^i(s^t),h_t^i(s^t)} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t^0(s^t) \left[\log\left(c_t^i(s^t)\right) - \left(1 - h_t^i(s^t)\right)^{(-1)}\right].$$

(c) Solve for individual consumption, hours and Arrow-Debreu prices (as a function of aggregate consumption)

Solution:

The Lagrangian for individual i reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t^0(s^t) \left[\log\left(c_t^i(s^t)\right) - \left(1 - h_t^i(s^t)\right)^{(-1)}\right] + \lambda^i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) \left[w^i h_t^i(s^t) - c_t^i(s^t)\right].$$

The optimality conditions of individual i with respect to labor supply, $h_t^i(s^t)$, read

$$0 = \frac{\partial \mathcal{L}}{\partial h_t^i(s^t)} = -\beta^t \pi_t^0(s^t) \left(1 - h_t^i(s^t)\right)^{-2} + \lambda^i q_t^0(s^t) w^i, \quad \forall s^t \in S^t,$$

Note that the Euler equation stated in Equation (1) still applies in this specification, but we are dealing with log-utility such that $\gamma = 1$. Combine the two optimality conditions for a given state s^t to yield

$$\beta^t \pi_t^0(s^t) \left(1 - h_t^i(s^t)\right)^{-2} = \lambda^i q_t^0(s^t) w^i = \beta^t \pi_t^0(s^t) c_t^i(s^t)^{-1} w^i, \quad \forall s^t \in S^t.$$

which can be reformulated as

$$c_t^i(s^t) = \left(1 - h_t^i(s^t)\right)^2 w^i, \quad \forall s^t \in S^t. \tag{6}$$

The same derivations as for Equation (4) apply and imply individual consumption growth will also be equalized across households.

$$c_t^i(s^t) = \gamma^i C_t(s^t), \quad \forall s^t \in S^t.$$

This implies that there is full risk sharing across states at the individual level in this economy. According to Equation (7), the Arrow-Debreu prices will be

$$q_t^0(s^t) = \frac{q_t^0(s^t)}{q_0^0(s)} = \beta^t \pi_t^0(s^t) \left(\frac{C_t(s^t)}{C_0(s^0)}\right)^{-\gamma}, \quad \forall s^t \in S^t. \tag{7}$$