Lecture 1 - Mechanism Design

Focus of first two lectures: what is the best way to allocate an object? We look at situations that are multi-agent extensions of the theory of screening.

This lecture looks at the most abstract setting, later we will focus on auction mechanisms. This lecture is based on K, chapter 5. More specifically, the mechanism we look at in this lecture need not be universal, that is, its rules can vary depending on the object for sale (via the distributions of buyers' values) nor it needs to be anonymous (we look at mechanisms that can treat different buyers differently). Auctions are instead anonymous and universal.

The allocation of an indivisible object is only one example of situations in which the design of mechanisms is a relevant tool. The example of the allocation of a private good captures a general theme: it is hard to find mechanisms compatible with individual incentives that ensure (a) efficient decisions, (b) voluntary participation of individuals, (c) balanced transfers.

Setting

A seller has an indivisible object to sell. The seller does not attach any value to the good. The seller is risk-neutral and wants to maximize the revenue from selling the good. There are N risk-neutral (potential) buyers, from set $\mathcal{N} = \{1, .., N\}$. Buyers have no budget constraints. Buyers have *private* and *independently distributed* values.[1] Buyer i's value X_i is distributed over the interval $\mathcal{X}_i := [0, \omega_i]$ according to CDF F_i with PDF f_i. We assume $f_i(x_i) > 0$ over the entire interval $\mathcal{X}_i$.

Let $\mathcal{X} := \times_{i=1}^{N} \mathcal{X}_i$ and $\mathcal{X}_{-i} := \times_{j \neq i} \mathcal{X}_j$. Let $f(x)$ be the joint density of $x := (x_1, .., x_N)$. Independence ensures

$$f(x) = f_1(x_1) \times f_2(x_2) \times ... \times f_N(x_N).$$

Define $f_{-i}(x_{-i})$ to be the joint density of $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, .., x_N)$.

A selling mechanism $(\mathcal{B}, \pi, \mu)$ is composed of

- a set of possible messages $\mathcal{B}_i$ for each buyer,

[1] Here "private" does not mean only privately known (although values ARE privately known). Private refers to the fact that buyer i's utility from the object depends only on her value X_i.

- an allocation rule $\pi : \mathcal{B} \to \Delta$, where Δ is the set of probability distributions over $\mathcal{N}$, (let $\pi_i(\mathbf{b})$ denote the probability that buyer i gets object (where $\mathbf{b} \in \mathcal{B}$))
- a payment rule $\mu : \mathcal{B} \to \mathbb{R}^N$ (let $\mu_i(\mathbf{b})$ denote the payment that i must make to the seller).

Buyers have quasi-linear preferences: individual i's utility is given by:

$$\pi_i(\mathbf{b})x_i - \mu_i(\mathbf{b}).$$

Every mechanism determines a game of incomplete information among the buyers. N strategies $\beta_i : [0, \omega_i] \to \mathcal{B}_i$ are part of an equilibrium of a mechanism if for all i and all x_i, given β_{-i}, $\beta_i(x_i)$ maximizes i's expected payoff. We look for Bayes Nash Equilibria. An allocation rule π is *efficient* if:

$$\sum_{\mathcal{N}} \pi_i(\mathbf{b})x_i \geq \sum_{\mathcal{N}} \pi'_i(\mathbf{b})x_i,$$

for any allocation π'.

The Revelation Principle.

Definition. *A mechanism is called a direct mechanism if $\mathcal{B}_i = \mathcal{X}_i$. We refer to a direct mechanism as $(\mathbf{Q}, \mathbf{M})$ where $Q_i(x)$ is the probability that i will get the object and $M_i(x)$ is the expected payment by i.*

In a direct mechanism, buyers are asked to simultaneously and independently report their types.

Proposition. *(**Revelation Principle**) Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his or her value truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.*

To see that the proposition holds, let $\mathbf{Q} : \mathcal{X} \to \Delta$ and $\mathbf{M} : \mathcal{X} \to \mathbb{R}^N$ be defined as: $\mathbf{Q}(x) = \pi(\beta(x))$ and $\mathbf{M}(x) = \mu(\beta(x))$ for all x.

Incentive Compatibility. For a direct mechanism $(\mathbf{Q}, \mathbf{M})$ let:

$$
\begin{aligned}
q_i(z_i) &= \int_{\mathcal{X}_{-i}} Q_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i}, \\
m_i(z_i) &= \int_{\mathcal{X}_{-i}} M_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i}.
\end{aligned}
$$

$q_i(z_i)$ is the conditional expected value of the probability that agent i obtains the good, conditioning on buyer i's reporting her type to be z_i. $m_i(z_i)$ is the conditional expected value of the transfer that agent i makes to the seller, conditioning on agent i reporting her type to be z_i.

Definition. *A direct mechanism is incentive compatible if* $\forall i \in \mathcal{N}$ *and* $\forall x_i \in \mathcal{X}_i$ *and* $\forall z_i \in \mathcal{X}_i$*:*

$$
U_i(x_i) := q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i).
$$

We call U_i *the equilibrium payoff function.*

Note that $q_i(z_i)x_i - m_i(z_i)$ is an affine (hence convex) function of x_i.

Hence, as incentive compatibility requires

$$
U_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\}
$$

then:

1) incentive compatibility implies convexity of $U_i(x_i)$ (the maximum over a set of convex functions is a convex function);

2) convex functions are not differentiable in at most countably many points. Consider any x_i for which $U_i(\cdot)$ is differentiable. Let $\delta > 0$. By incentive compatibility we have:

$$
\begin{aligned}
\lim_{\delta \to 0} \frac{U_i(x_i + \delta) - U_i(x_i)}{\delta} &\geq \lim_{\delta \to 0} \frac{(x_i + \delta)U_i(x_i) - m_i(x_i) - (x_i U_i(x_i) - m_i(x_i))}{\delta} = q_i(x_i), \\
\lim_{\delta \to 0} \frac{U_i(x_i) - U_i(x_i - \delta)}{\delta} &\leq \lim_{\delta \to 0} \frac{(x_i)U_i(x_i) - m_i(x_i) - ((x_i - \delta)\, U_i(x_i) - m_i(x_i))}{\delta} = q_i(x_i),
\end{aligned}
$$

thus

$$U_i'(x_i) = q_i(x_i);$$

3) as any continuous function is the definite integral of its derivative, we have:

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i)dt_i. \tag{1}$$

Using the definition of $U_i(x_i)$, (1) can be used to characterize the conditional expected transfer $m_i(x_i)$:

$$\begin{aligned} U_i(x_i) = \quad q_i(x_i)x_i - m_i(x_i) = & \quad -m_i(0) + \int_0^{x_i} q_i(t_i)dt_i \\ \leftrightarrow & \\ m_i(x_i) = & \qquad q_i(x_i)x_i + m_i(0) - \int_0^{x_i} q_i(t_i)dt_i \end{aligned} \tag{2}$$

As the *shape* of the payoff function is entirely determined by the allocation rule Q, the payment rule is determined by the IC constraint.

4) If a direct mechanism is incentive compatible, then $q_i(\cdot)$ is non-decreasing.

To see this, consider 2 types x_i, z_i such that $x_i > z_i$. Incentive compatibility requires:

$$\begin{aligned} q_i(x_i)x_i - m_i(x_i) &\geq q_i(z_i)x_i - m_i(z_i), \\ q_i(z_i)z_i - m_i(z_i) &\geq q_i(x_i)z_i - m_i(x_i). \end{aligned}$$

Subtracting the two inequalities: $q_i(x_i)x_i - m_i(x_i) - (q_i(x_i)z_i - m_i(x_i)) \geq q_i(z_i)x_i - m_i(z_i) - (q_i(z_i)z_i - m_i(z_i)) \leftrightarrow q_i(x_i)(x_i - z_i) \geq q_i(z_i)(x_i - z_i) \leftrightarrow q_i(x_i) \geq q_i(z_i)$.

Individual Rationality. In many (but not all) applications, it makes sense to assume that potential buyers participate in the mechanism *after* learning their types, so individual rationality requires that for all i and x_i we have $U_i(x_i) \geq 0$.

Definition. *A direct mechanism is individually rational if* $\forall i \in \mathcal{N}$ *and* $\forall x_i \in \mathcal{X}_i$ *:*

$$U_i(x_i) \geq 0.$$

By equation (1) we see that it is sufficient to check that $U_i(0) \geq 0$ (and since $U_i(0) = -m_i(0)$, this is equivalent to $m_i(0) \leq 0$).

Optimal Mechanisms

Optimal mechanisms = mechanisms that maximize the seller expected revenue, subject to IC and IR constraints. We focus on direct mechanisms.

The expected revenue of seller is:

$$E[R] = \sum_{i \in \mathcal{N}} E[m_i(X_i)]$$

ex-ante expected payment of buyer i:

$$E[m_i(X_i)] \;=\; \int_0^{\omega_i} m_i(x_i) f_i(x_i) dx_i$$

Using (2):

$$E[m_i(X_i)] \;=\; m_i(0) + \int_0^{\omega_i} q_i(x_i) x_i f_i(x_i) dx_i - \int_0^{\omega_i} \int_0^{x_i} q_i(t_i) f_i(x_i) dt_i dx_i.$$

Switching the order of integration, the last term can be written:

$$\begin{aligned}
\int_0^{\omega_i} \int_0^{x_i} q_i(t_i) f_i(x_i) dt_i dx_i &= \int_0^{\omega_i} \int_{t_i}^{\omega_i} q_i(t_i) f_i(x_i) dx_i dt_i \\
&= \int_0^{\omega_i} q_i(t_i) \int_{t_i}^{\omega_i} f_i(x_i) dx_i dt_i \\
&= \int_0^{\omega_i} q_i(t_i)(1 - F_i(t_i)) dt_i
\end{aligned}$$

So:

$$
\begin{aligned}
E\left[m_i(X_i)\right] &= m_i(0) + \int_0^{\omega_i} q_i(x_i) x_i f_i(x_i) dx_i - \int_0^{\omega_i} q_i(x_i)(1 - F_i(x_i)) dx_i \\
&= m_i(0) + \int_0^{\omega_i} q_i(x_i) f_i(x_i) \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) dx_i
\end{aligned}
$$

Using $q_i(x_i) = \int_{\mathcal{X}_{-i}} Q_i(x) f_{-i}(x_{-i}) dx_{-i}$, and $(f(x) = f_1(x_1) \times f_2(x_2) \times \ldots \times f_N(x_N))$, we have:

$$
\begin{aligned}
E\left[m_i(X_i)\right] &= m_i(0) + \int_0^{\omega_i} \int_{\mathcal{X}_{-i}} Q_i(x) f(x) dx_{-i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) dx_i \\
&= m_i(0) + \int_{\mathcal{X}} Q_i(x) f(x) \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) dx
\end{aligned}
$$

Where we are using:

$$
\int_{\mathcal{X}} g(x) dx = \int_{x_1} \int_{x_2} .. \int_{x_n} g(x) dx_n .. dx_2 dx_1 .
$$

The objective of the seller is to find a mechanism that maximizes

$$
\sum_{i \in \mathcal{N}} E\left[m_i(X_i)\right] = \sum_{i \in \mathcal{N}} m_i(0) + \sum_{i \in \mathcal{N}} \int_{\mathcal{X}} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(x) f(x) dx .
$$

subject to IC and IR constraint.

Definition. *Let $\psi_i(x_i) := x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}$ be called the virtual valuation of a buyer with value x_i. A design problem is regular if for all i function ψ_i is increasing in the true value x_i.*

Note that since $\psi_i(x_i) := x_i - \frac{1}{\lambda_i(x_i)}$,where $\lambda_i := \frac{f_i}{1 - F_i}$ is the hazard rate of F_i, a sufficient condition for regularity is that λ_i is non-decreasing (λ_i is non-decreasing for the uniform, normal, exponential distributions, among others). From now on, we assume regularity.

The function that the seller maximizes can be re-written as:

(3)
$$
\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} \sum_{i \in \mathcal{N}} \left(\psi_i(X_i) Q_i(x) \right) f(x) dx .
$$

Hence we observe:

$$Q_i(x) > 0 \leftrightarrow \psi_i(x_i) = \max_{j \in \mathcal{N}} \psi_j(x_j) \geq 0 \tag{4}$$

to the object is assigned with probability 1 to one of the buyers with the highest virtual valuation, as long as the virtual valuation is non-negative.

$$M_i(x) = Q_i(x)x_i - \int_0^{x_i} Q_i(z_i, x_{-i})dz_i \tag{5}$$

Conditions (4) and (5) together define an optimal mechanism. To get a more intuitive interpretation:

$$y_i(x_{-i}) = \inf \{z_i : \psi_i(z_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(z_i) \geq \psi_j(x_j)\}$$

thus (4) is equivalent to:

$$Q_i(z_i, x_{-i}) = \begin{cases} 1 & \text{if } z_i > y_i(x_{-i}) \\ 0 & \text{if } z_i < y_i(x_{-i}) \end{cases}$$

thus

$$\int_0^{x_i} Q_i(z_i, x_{-i})dz_i = \begin{cases} x_i - y_i(x_{-i}) & \text{if } x_i > y_i(x_{-i}) \\ 0 & \text{if } x_i < y_i(x_{-i}) \end{cases}$$

and therefore:

$$M_i(x) = \begin{cases} y_i(x_{-i}) & \text{if } Q_i(x) = 1 \\ 0 & \text{if } Q_i(x) = 0 \end{cases}$$

This mechanism is not efficient:

1) the mechanism requires the seller to keep the good if all virtual valuations are negative, but as all buyers have positive valuations and seller has valuation 0, efficiency requires that the seller assigns the good to the seller with the highest valuation rather than keeping it.
2) the object is allocated to the buyer with the highest virtual valuation, and this needs not be the agent with the highest value (though the two correspond in case of identical distributions of valuations).
In the setup considered so far, a second-price auction is efficient: in a second-price auction it is optimal for every buyer to report her type truthfully, and the object is allocated to one of the buyers with the highest report, so with the highest valuation.

Symmetric Example

Suppose $N = 2$, $\omega_i = 1$, X_i is distributed uniformly, for $i = 1, 2$.
Thus: $\psi_i(x_i) = x_i - \frac{1-F(x_i)}{x_i} = 2x_i - 1$. Note that the design problem is regular.
The revenue maximizing mechanism does not sell to any buyer if for both $i = 1$ and $i = 2$:
$\psi_i(x_i) < 0 \leftrightarrow x_i < \frac{1}{2}$.
If the good is sold, it is sold to the buyer with the highest virtual valuation, that is, with the buyer with the highest valuation.
It can be checked that a first or second price auction with reverve price $\frac{1}{2}$ will implement this mechanism (not that in this case the efficient mechanism is a first or a second price auction with reserve price 0).

Asymmetric Example

Suppose $N = 2$, $\omega_1 = \omega_2 = 1$, and $F_1(x_1) = x_1^2$, $F_2(x_2) = 2x_2 - x_2^2$.
Thus $\psi_1(x_1) = \frac{3}{2}x_1 - \frac{1}{2x_1}$ and $\psi_2(x_2) = \frac{3}{2}x_2 - \frac{1}{2}$.
The revenue maximizing mechanism does not sell to any buyer if
$\psi_1(x_1) < 0 \leftrightarrow x_1 < \sqrt{\frac{1}{3}}$; $\psi_2(x_2) < 0 \leftrightarrow x_1 < \frac{1}{3}$.
If the good is sold, it is sold to buyer 1 if $\psi_1(x_1) > \psi_2(x_2) \leftrightarrow x_2 < x_1 + \frac{1}{3} - \frac{1}{3x_1}$

Efficient Mechanisms

Consider a more general setup in which $\mathcal{X}_i = [\alpha_i, \omega_i] \subset \mathbb{R}$ for each agent.

An allocation rule $Q^* : \mathcal{X} \to \Delta$ is efficient if for all $x \in \mathcal{X}$,

$$Q^*(x) \in \arg\max_{Q \in \Delta} \sum_{j \in \mathcal{N}} Q_j x_j.$$

That is, an efficient allocation rule allocates the object to one of the agents with the highest valuation.

The maximized value of the social welfare is defined as:

$$W(x) = \sum_{j \in \mathcal{N}} Q_j^*(x) x_j.$$

The VCG Vickrey-Clarke-Groves Mechanism. The VCG mechanism is an efficient mechanism (Q^*, M^V) where:

$$M_i^V(x) = W(\alpha_i, x_{-i}) - W_{-i}(x)$$

for

$$W_{-i}(x) = \sum_{j \neq i} Q_j^*(x) x_j.$$

With $\alpha_i = 0$, the VCG mechanism corresponds to a second price auction. Note that the mechanism can be thought of as a "pivotal mechanism":

- Say for reports x the object is not assigned to agent i. Then $W(\alpha_i, x_{-i}) = W_{-i}(x)$, thus $M_i^V(x) = 0$
- Say for reports x the object is assigned to agent i with some positive probability. Then $W(\alpha_i, x_{-i}) < W_{-i}(x)$, thus $M_i^V(x) < 0$.

Thus an agent pays only if she is pivotal, that is only if her presence changes the utility of the other agents. In this case, the payment is exactly equal to the loss of value imposed by the agent on the other agents.

The VCG is incentive compatible. As long as others report x_{-i}, by reporting z_i buyer $i's$ payoff is:

$$Q_i^*(z_i, x_{-i})x_i - M^V(z_i, x_{-i}) = \sum_{j \in \mathcal{N}} Q_j^*(z_i, x_{-i})x_j - W(\alpha_i, x_{-i})$$

Note that the term $\sum_{j \in \mathcal{N}} Q_j^*(z_i, x_{-i})x_j$ is maximized by reporting $z_i = x_i$, while the term $-W(\alpha_i, x_{-i})$ does not depend on the report, so reporting truthfully is optimal.

Using what we learned, as the mechanism is incentive compatible, we know that the equilibrium payoff:

$$U_i^V(x_i) = E\left[W(x_i, X_{-i}) - W(\alpha_i, X_{-i})\right]$$

is convex and increasing. Hence $U_i^V(\alpha_i) = 0$ implies that IR holds.

Two observations:

1) Any mechanism that satisfies IC and IR and is efficient must have an M function with the same shape as the VCG mechanism. Moreover $U_i^V(\alpha_i) = 0$ ensures that the VCG mechanism maximizes expected payments among the efficient mechanisms that satisfy IC and IR.

2) A mechanism balances the budget (ex-post) if for all x:

$$\sum_{\mathcal{N}} M_i(x) = 0.$$

VCG is not budget balanced *ex post.*

An example of efficient mechanism that satisfies ex post budget balance is the Arrow- d'Aspremont-Gerard-Varet (or AGV) mechanism (Q^*, M^A), defined by:

$$M_i^A(x) = \frac{1}{N-1} \sum_{j \neq i} \left(E_{x_{-j}}\left[W_{-j}(x_j, X_{-j})\right] \right) - E_{x_{-i}}\left[W_{-i}(x_i, X_{-i})\right].$$

So:

$$\sum_{\mathcal{N}} M_i^A(x) = \sum_i \left(\frac{1}{N-1} \sum_{j \neq i} \left(E_{x_{-j}} \left[W_{-j}(x_j, X_{-j}) \right] \right) - E_{x_{-i}} \left[W_{-i}(x_i, X_{-i}) \right] \right).$$

$$= \textstyle\sum_i \left(\frac{1}{N-1} (N-1) \left(E_{x_{-i}} \left[W_{-i}(x_i, X_{-i}) \right] \right) - E_{x_{-i}} \left[W_{-i}(x_i, X_{-i}) \right] \right) = 0.$$

The AGV mechanism is incentive compatible. To see this, suppose all other agents are reporting x_{-i} truthfully. The expected payoff to i from reporting z_i when the true value is x_i is equal to:

$$E_{x_{-i}} \left[Q_i^*(z_i, X_{-i}) x_i + W_{-i}(z_i, X_{-i}) \right] - E_{x_{-i}} \left[\frac{1}{N-1} \sum_{j \neq i} E_{x_{-j}} \left[W_{-j}(x_j, X_{-j}) \right] \right].$$

The second term is independent of z_i, while the first term is maximized by $z_i = x_i$. On the other hand, the AGV mechanism may not satisfy the IR constraint.